

# LPC(ID): A Sequent Calculus Proof System for Propositional Logic Extended with Inductive Definitions

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**Abstract.** The logic FO(ID) uses ideas from the field of logic programming to extend first order logic with non-monotone inductive definitions. Such logic formally extends logic programming, abductive logic programming and datalog, and thus formalizes the view on these formalisms as logics of (generalized) inductive definitions. The goal of this paper is to study a deductive inference method for PC(ID), which is the propositional fragment of FO(ID). We introduce a formal proof system based on the sequent calculus (Gentzen-style deductive system) for this logic. As PC(ID) is an integration of classical propositional logic and propositional inductive definitions, our sequent calculus proof system integrates inference rules for propositional calculus and definitions. We present the soundness and completeness of this proof system with respect to a slightly restricted fragment of PC(ID). We also provide some complexity results for PC(ID). By developing the proof system for PC(ID), it helps us to enhance the understanding of proof-theoretic foundations of FO(ID), and therefore to investigate useful proof systems for FO(ID).

## 1 Introduction

In this paper, we study deductive methods for the propositional fragment of FO(ID) [14]. To motivate this study, we need to say a few words about the origin and the motivation of FO(ID).

Perhaps the two most important knowledge representation paradigms of the moment are on the one hand, classical logic-based approaches such as description logics [2], and on the other hand, rule-based approaches based on logic programming and extensions such as Answer Set Programming and Abductive Logic Programming [3,24]. The latter disciplines are rooted firmly in the discipline of Non-Monotonic Reasoning [32]. FO(ID) integrates both paradigms in a tight, conceptually clean manner. The key to integrate “rules” into classical logic (FO) is the observation that natural language, or more precisely, the informal language of mathematicians, has an informal rule-based construct: the construct of *inductive/recursive definitions* (IDs).

In Figure 1 and Figure 2, we displayed two prototypical examples of the most common forms of inductive definitions in mathematics: monotone ones, respectively definitions by induction over a well-founded order. As seen in these figures, both

**Definition 1.** The transitive closure  $T_G$  of a directed graph  $G$  is defined by induction:

- $(x, y) \in T_G$  if  $(x, y) \in G$ ;
- $(x, y) \in T_G$  if for some vertex  $z$ ,  $(x, z) \in T_G$  and  $(z, y) \in T_G$ .

**Fig. 1.** Definition of Transitive closure

**Definition 2.** The satisfaction relation  $\models$  between  $\sigma$ -interpretations  $I$  and propositional formulas over  $\sigma$  is defined by structural induction:

- $I \models p$  if  $p$  is an atom and  $p \in I$ ,
- $I \models \psi \wedge \phi$  if  $I \models \psi$  and  $I \models \phi$ ,
- $I \models \psi \vee \phi$  if  $I \models \psi$  or  $I \models \phi$ ,
- $I \models \neg\psi$  if  $I \not\models \psi$ .

**Fig. 2.** Definition of satisfaction

are frequently represented as a set of informal rules. These two forms of inductive definitions are generalized by the concept of *iterated* inductive definitions (IID) [6]. Inductive definitions define their concept by describing how to *construct* it through a process of iterated application of rules starting from the empty set. Definitions by induction over a well-founded order are frequently non-monotone, as illustrated by the non-monotone rule “ $I \models \neg\psi$  if  $I \not\models \psi$ ” which derives the satisfaction of  $\neg\psi$  given the non-satisfaction of  $\psi$ .

Of course, a definition is not just a set of material implications. Thus, a sensible scientific research question is to design a uniform, rule-based formalism for representing these forms of definitions. Such a study is not only useful as a formal logic study of the concept of inductive definition but it contributes to the understanding of rule-based systems and thus, to the study of the (formal and informal) semantics of logic programming and the integration of classical logic-based and rule-based approaches to knowledge representation.

Iterated inductive definitions have been studied in mathematical logic [6] but the formalisms there are not rule-based and require an extremely tedious encoding of rules and well-founded orderings into one complex formula [14]. In several papers [11,12,14], it was argued that, although unintended by its inventors, the rule-based formalism of logic programming under the well-founded semantics [20] and its extension to rules with FO-bodies in [19] correctly formalizes the above mentioned forms of inductive definitions. Stated differently, if we express an informal inductive definition of one of the above kinds into a set of formal rules

$$\forall \bar{x} (P(\bar{t}) \leftarrow \phi)$$

then the informal semantics of the original definition matches the well-founded semantics of the formal rule set. E.g., in a well-founded model of the following “literal” translation of the definition in Figure 1:

$$\left\{ \begin{array}{l} \forall x, y (T_G(x, y) \leftarrow G(x, y)) \\ \forall x, y (T_G(x, z) \leftarrow (\exists z T_G(x, y) \wedge T_G(y, z))) \end{array} \right\}$$

$T_G$  is interpreted as the transitive closure of the graph interpreting  $G$ . A similar claim holds for the literal translation of the definition of  $\models$  in Figure 2. Thus, the rule formalism under the well-founded semantics provides the desired uniform syntax and semantics for representing the above mentioned forms of inductive definition construct.

There are several good arguments to integrate the above inductive definition construct (and hence, this generalized form of logic programming under the well-founded semantics) into FO. (1) FO and definitions are complementary KR languages: FO is a base language very suitable for expressing propositions, assertions or constraints while it is well-known that, in general, inductive definitions cannot be expressed in FO [25]. (2) Definitions are important for KR. In the case of *non-inductive* definitions, their use for defining terminology was argued long time ago in Brachman and Levesque’s seminal paper [5] and was the motivation for developing description logics [2]. As for *inductive* definitions, they are quite likely as important to declarative Knowledge Representation as recursive functions and procedures are to programming. Applications of inductive definitions abound in KR: various instances of transitive closure, definitions of recursive types and of concepts defined over recursive types, descriptions of dynamic worlds through definitions of states in terms of past states and effects of actions, etc. In [13], a formalization of situation calculus in terms of iterated inductive definitions in FO(ID) yields an elegant and very general solution for the ramification problem in the context of the situation calculus. (3) Inductive definitions are also an interesting *Non-Monotonic Reasoning* language construct. A logic is non-monotonic if adding new expressions to a theory may invalidate previous inferences. Obviously, adding a new rule to an inductive definition defines a different set and hence, this operation may invalidate previous inferences<sup>3</sup>. One of the main non-monotonic reasoning principles is the Closed World Assumption (CWA) [35]. The intuition underlying CWA is that “an atom is false unless it can be proven”. This matches with an inductive definition in which a defined atom  $P(\bar{t})$  is false unless it is explicitly derived by one of its rules  $P(\bar{t}) \leftarrow \psi$  during the construction process. Hence, inductive definitions can be viewed as a very precise and well-understood form of Closed World Assumption. Moreover, it is well-known that rule formalisms under CWA can be used to represent many useful forms of *defaults*. The correspondence between CWA and inductive definition construct implies that the methodologies to represent defaults developed in, e.g., logic programming, can be used in an inductive definition formalism as well. Domain Closure [31] is another important non-monotonic reasoning principle that can be expressed through inductive definitions [14].

All the above provides a strong motivation for adding inductive definitions to FO. Thus, the resulting logic FO(ID) extends FO not only with an inductive definition construct but also with an expressive and precise non-monotonic reasoning principle. Not surprisingly, the logic FO(ID) is strongly tied to many other logics. It is an extension of FO with inductive definitions and a conceptually clean integration of FO and LP. It integrates monotonic and non-monotonic logics. The

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<sup>3</sup> Observe that the concept of (non-)monotonicity is used here in two different ways. Adding a rule to a *monotone inductive definition* is a *non-monotonic reasoning* operation.

inductive definition construct of FO(ID) formally generalizes Datalog [1]: this is a natural match, given that Datalog programs aim to *define* queries and views. FO(ID) is also strongly related to fixpoint logics. Monotone definitions in FO(ID) are a different -rule-based- syntactic sugar of the fixpoint formulas of Least Fixpoint Logic (LFP) [33,34]. Last but not least, FO(ID), being a conceptually clean, well-founded integration of rules into classical logic, might play a unifying role in the current attempts of extending FO-based description logics with rules [38]. It thus appears that FO(ID) occupies quite a central position in the spectrum of computational and knowledge representation logics.

Several attempts to build inference systems for FO(ID) are underway. One line of research is the development of finite model generators [28,27,29,39] . They have similar applications and speed as current Answer Set Programming solvers [17,16]. However, in this paper we study a more traditional form of inference: deduction. As for every formal logical system, the development of deductive inference methods for FO(ID) is an important research topic. There is no hope of course to build a complete proof system of FO(ID). Indeed, inductive definability leads to undecidability, not even semi-decidability. As such, the task we set out for this paper is restricted to the development of a sound proof system and a decidable fragment of FO(ID).

The goal of this paper is to extend the propositional part of Gentzen’s sequent calculus to obtain a proof system for PC(ID), the propositional fragment of FO(ID). We view our work as an initial investigation to build proof systems for (fragments of) FO(ID). In proof theory, Gentzen’s sequent calculus **LK** [21,36] is a widely known proof system for first order logic. The sequent calculus is well-suited to a goal-directed approach for constructing logical derivations. The advantage of the method is its theoretical elegance and the fact that it focuses the proof search, with applicable proof rules constrained by logical connectives appearing in the current goal sequent. Our work is inspired by the one of Compton, who used sequent calculus (Gentzen-style deductive system) methods in [7,8] to investigate sound and complete deductive inference methods for existential least fixpoint logic and stratified least fixpoint logic. Existential least fixpoint logic, as described in [7], is a logic with a least fixpoint operator but only existential quantification and stratified least fixpoint logic, as shown in [8], is a logic with a least fixpoint operator and characterizes the expressibility of stratified logic programs. Indeed, these two logics without nested least fixpoint expressions can be viewed as fragments of FO(ID).

The contributions of this paper can be summarized as follows:

1. We introduce a sequent calculus for PC(ID).
2. We prove that the deductive system is sound and complete for a slightly restricted fragment of PC(ID).
3. We provide some complexity results for PC(ID).

By developing a proof system for PC(ID), we want set a first step to enhance the understanding of proof-theoretic foundations of FO(ID). One application of this work could be for the development of tools to check the correctness of the outputs generated by PC(ID) model generators such as MINISAT(ID) [29]. Given a PC(ID) theory  $T$  as input, such a solver outputs a model for  $T$  or concludes that

$T$  is unsatisfiable. In the former case, an independent *model checker* can be used to check whether the output is indeed a model of  $T$ . However, when the solver concludes that  $T$  is unsatisfiable, it is less obvious how to check the correctness of this answer. One solution is to transform a trace of the solvers computation into a proof of unsatisfiability in a PC(ID) proof system. An independent *proof checker* can then be used to check this formal proof. Model and proof checkers can be a great help to detect bugs in model generators. An analogous checker for the Boolean Satisfiability problem (SAT) solvers was described in [40].

On the longer run, we view our work also as a first step towards the development of proof systems and decidable fragments of FO(ID). A potential use of this is in the field of description logics. Deductive reasoning is the distinguished form of inference of Description Logics. Given the efforts to extend Description Logics with rules and the fact that FO(ID) offers a natural, clean integration of a very useful form of rules in FO, it seems that research on decidable fragments of FO(ID) could play a useful role in that area.

The structure of this paper is as follows. We introduce PC(ID) in Section 2. We present a sequent calculus proof system for PC(ID) in Section 3. The main results of the soundness and completeness of the proof system are investigated in Section 4. We provide some complexity results for PC(ID) in Section 5. We finish with conclusions, related and future work.

## 2 Preliminaries

In this section, we present PC(ID), which is the propositional fragment of FO(ID) [14]. Observe that PC(ID) is an extension of propositional calculus (PC) with propositional inductive definitions (IDs).

### 2.1 Syntax

A propositional vocabulary  $\tau$  is a set of propositional atoms. A formula of propositional calculus over  $\tau$ , or briefly, a PC-formula over  $\tau$ , is inductively defined as:

- an atom in  $\tau$  is a PC-formula over  $\tau$ ;
- if  $F$  is a PC-formula over  $\tau$ , then so is  $\neg F$ ;
- if  $F_1, F_2$  are PC-formulas over  $\tau$ , then so are  $F_1 \wedge F_2$  and  $F_1 \vee F_2$ .

We use the following standard abbreviations:  $F_1 \supset F_2$  for  $\neg F_1 \vee F_2$  and  $F_1 \equiv F_2$  for  $(F_1 \wedge F_2) \vee (\neg F_1 \wedge \neg F_2)$ . A *literal* is an atom  $P$  or its negation  $\neg P$ . An atom  $P$  has a *negative (positive) occurrence* in formula  $F$  if  $P$  has an occurrence in the scope of an odd (even) number of occurrences of the negation symbol  $\neg$  in  $F$ .

A *definition*  $D$  over  $\tau$  is a finite set of rules of the form:

$$P \leftarrow \varphi,$$

where  $P \in \tau$  and  $\varphi$  is a PC-formula over  $\tau$ . Note that the symbol “ $\leftarrow$ ” is a new symbol, which must be distinguished from (the inverse of) material implication  $\supset$ . For a rule of the above form, the atom  $P$  is called the *head* of the rule while  $\varphi$

is known as its *body*. An atom appearing in the head of a rule of  $D$  is called a *defined* atom of  $D$ , any other atom is called an *open* atom of  $D$ . We denote the set of defined atoms by  $\tau_D^d$  and that of all open ones by  $\tau_D^o$ . We call a definition  $D$  *positive* if its defined symbols have only positive occurrences in rule bodies (i.e., occur in the scope of an even number of negation symbols).

$D$  is called *inductive* or *recursive* in predicate  $P$  if its dependency relation  $\prec$  satisfies  $P \prec P$ . Here, the dependency relation  $\prec$  of  $D$  on  $\tau$  is the transitive closure of the set of all pairs  $(Q, P)$  such that for some rule  $P \leftarrow \varphi \in D$ ,  $Q$  occurs in  $\varphi$ . The intended *informal semantics* of a formal definition  $D$  is given by understanding it as a -possibly inductive- definition of the defined symbols in terms of the open symbols. This understanding is clear in case of positive definitions and the corresponding formal semantics is obvious. In the next sections, we consider how this view extends to arbitrary non-positive definitions.

A PC(ID)-formula over  $\tau$  is defined by the following induction:

- an atom in  $\tau$  is a PC(ID)-formula over  $\tau$ ;
- a definition over  $\tau$  is a PC(ID)-formula over  $\tau$ ;
- if  $F$  is a PC(ID)-formula over  $\tau$ , then so is  $\neg F$ ;
- if  $F_1, F_2$  are PC(ID)-formulas over  $\tau$ , then so are  $F_1 \wedge F_2$  and  $F_1 \vee F_2$ .

A PC(ID) theory over  $\tau$  is a set of PC(ID)-formulas over  $\tau$ .

Any definition containing multiple rules with the same atom in the head can be easily transformed into a definition with only one rule per defined atom. We illustrate this by the following example.

*Example 1.* The following definition

$$\left\{ \begin{array}{l} P \leftarrow O_1 \wedge Q \\ P \leftarrow P \\ Q \leftarrow Q \wedge P \\ Q \leftarrow O_2 \end{array} \right\}$$

is equivalent to this one:

$$\left\{ \begin{array}{l} P \leftarrow (O_1 \wedge Q) \vee P \\ Q \leftarrow (Q \wedge P) \vee O_2 \end{array} \right\}.$$

As we mentioned in Section 1, monotone definitions in FO(ID) are a different -rule-based- syntactic sugar of the fixpoint formulas of Least Fixpoint Logic (LFP). We now illustrate the relation between a propositional inductive definition and a propositional least fixpoint expression in fixpoint logics.

A *propositional least fixpoint expression* is of the form:

$$[LFP_{P_1, \dots, P_n}(\theta_1, \dots, \theta_n)]\psi,$$

where for each  $i \in [1, \dots, n]$ ,  $P_i$  is a propositional atom,  $\theta_i$  is either a propositional formula or a propositional least fixpoint expression,  $\psi$  is either a propositional formula or a propositional least fixpoint expression, and  $P_i$  occurs only positively in  $\theta_i$  and  $\psi$ . Note that the subformulas  $\psi, \theta_1, \dots, \theta_n$  of a least fixpoint expression  $[LFP_{P_1, \dots, P_n}(\theta_1, \dots, \theta_n)]\psi$  may contain least fixpoint expressions. Indeed, nesting

of least fixpoint expressions is allowed in fixpoint logics. But nesting of definitions is not allowed in PC(ID). All subformulas  $\psi, \theta_1, \dots, \theta_n$  of an unnested least fixpoint expression contain only positive occurrences of each atom  $P_i$ . It is worth mentioning that the unnested least fixpoint expression  $[LFP_{P_1, \dots, P_n}(\theta_1, \dots, \theta_n)]\psi$ , where  $\theta_1, \dots, \theta_n, \psi$  may not contain least fixpoint expressions, corresponds exactly to the second order PC(ID)-formula

$$\exists P_1 \dots P_n \left( \left\{ \begin{array}{c} P_1 \leftarrow \theta_1 \\ \vdots \\ P_n \leftarrow \theta_n \end{array} \right\} \wedge \psi \right).$$

However, such a correspondence does not hold for nested least fixpoint expressions since only PC-formulas are allowed as bodies of rules in definitions.

In summary, the differences between the definition construct and the fixpoint definitions are:

- The fixpoint notation is formula-based and defines predicate variables with scope restricted to the fixpoint expression while a definition construct is rule-based and defines predicate constants. (These are “syntactic sugar” differences.)
- Fixpoint expressions can be nested while definitions cannot. On the other hand, in fixpoint expressions, the defined variables can occur only positively in the defining formulas, while in definitions, the defined predicates can occur negatively in rule bodies.

The relation between definitions and LFP are investigated in [23].

## 2.2 Semantics

In this section, we formalize the informal semantics of the two most common forms of inductive definition, monotone inductive definitions (e.g., the definition of transitive closure, Figure 1) and definitions over a well-founded order (e.g., the definition of the satisfaction relation  $\models$ , Figure 2), and their generalization, the notion of an iterated inductive definition. These informal types of definitions might be roughly characterized as follows:

- The rules of a monotone inductive definition of a set add objects to the defined set given the *presence* of certain other objects in the set.
- For an inductive definition over some (strict) well-founded order, a rule adds an object  $x$  given the *presence* or *absence* of certain other *strictly smaller* objects in the set.
- Finally, an iterated inductive definition is associated with a well-founded semi-order<sup>4</sup> such that each rule adds an object  $x$  given the *presence* of some other *less or equivalent* objects in the defined set and the *absence* of some *strictly less* objects.

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<sup>4</sup> A semi-order  $\leq$  is a transitive reflexive binary relation. Two elements  $x, y$  are  $\leq$ -equivalent if  $x \leq y$  and  $y \leq x$ , and  $x$  is strictly less than  $y$  if  $x \leq y$  and  $y \not\leq x$  are not equivalent. A semi-order is well-founded if it has no infinite strictly descending chains  $x_0 > x_1 > x_2 > \dots$ .

According to this characterizations, iterated inductive definitions generalize the other types. Non-monotonicity of the two latter types of definitions stem from rule conditions that refer to the absence of objects in the defined set (as in the condition of “ $I \models \neg\psi$  if  $I \not\models \psi$ ”). Adding a new element to the set might violate a condition what was previously satisfied. For an extensive argument how the well-founded semantics uniformly formalizes these three principles, we refer to [12,14]. Below, we just sketch the main intuitions.

As we all know, the set defined by any of the aforementioned forms of inductive definitions can be obtained constructively as the limit of an increasing sequence of sets, by starting with the empty set and iteratively applying unsatisfied rules until saturation. A key difference between monotone definitions and non-monotone inductive definitions is that in the first, once the condition of a rule is satisfied in some intermediate set, it holds in all later stages of the construction. This is not the case for non-monotone inductive definitions. E.g., in the construction of  $\models$ , the set of formulas  $\psi$  for which the condition of the rule “ $I \models \neg\psi$  if  $I \not\models \psi$ ” holds, initially contains all formulas and gradually decreases. As a consequence, the order of rule applications is arbitrary for monotone inductive definitions but matters for non-monotone definitions. There, it is critical to delay application of an unsatisfied rule until it is certain that its condition will not be falsified by later rule applications. This is taken care of by applying the rules along the well-founded order provided with the definition (e.g., the subformula order in the definition of  $\models$ ). In particular, application of a rule deriving some element  $x$  is delayed until no unsatisfied rule is left deriving a strictly smaller object  $y < x$ .

It would be rather straightforward to formalize this idea for PC(ID) if it was not that a PC(ID) definition  $D$  does not come with a explicit order. Fortunately, there is a different way to make sure that a rule can be safely applied, i.e., that later rule applications during the inductive process will not falsify its condition. To do this, we need to distinguish whether a defined atomic proposition has been derived to be true, to be false or is still underived. E.g., once  $I \models \psi$  is derived to be true, we can safely apply the rule for disjunctions and derive  $I \models \psi \vee \phi$  to be true, even  $I \models \phi$  is still underived. Likewise, we can safely derive  $I \not\models \psi \wedge \phi$  as soon as we found out  $I \not\models \psi$ . Applying this criterion relies on the ability to distinguish whether a defined atomic proposition (such as “ $I \models \psi$ ”) has been derived to be true, to be false or is still underived, and whether a rule condition is certainly satisfied, certainly dissatisfied or still unknown in such state. This naturally calls for a formalization of the induction process in a three-valued setting where intermediate stages of the set in construction are represented by three-valued sets instead of two-valued sets, and rules are evaluated in these three-valued sets.

Below we present the formalization of the well-founded semantics introduced in [15]. Compared to the original formalizations in [20,19], it is geared directly at formalizing the inductive process as described above, using concepts of three-valued logic. We start its presentation by recalling some basic concepts of three-valued logic.

Consider the set of truth values  $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ . The *truth order*  $\leq$  on this set is induced by  $\mathbf{f} \leq \mathbf{u} \leq \mathbf{t}$  and the *precision order*  $\leq_p$  is induced by  $\mathbf{u} \leq_p \mathbf{f}$  and  $\mathbf{u} \leq_p \mathbf{t}$ . Define  $\mathbf{f}^{-1} = \mathbf{t}$ ,  $\mathbf{u}^{-1} = \mathbf{u}$  and  $\mathbf{t}^{-1} = \mathbf{f}$ .



Let  $\tau$  be a propositional vocabulary. A three-valued  $\tau$ -interpretation, also called a  $\tau$ -valuation, is a function  $I$  from  $\tau$  to the set of truth values  $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ . An interpretation is called two-valued if it maps no atoms to  $\mathbf{u}$ . Given two disjoint vocabularies  $\tau$  and  $\tau'$ , a  $\tau$ -interpretation  $I$  and a  $\tau'$ -interpretation  $I'$ , the  $\tau \cup \tau'$ -interpretation mapping each element  $P$  of  $\tau$  to  $I(P)$  and each  $P \in \tau'$  to  $I'(P)$  is denoted by  $I + I'$ . When  $\tau' \subseteq \tau$ , we denote the restriction of a  $\tau$ -interpretation  $I$  to the symbols of  $\tau'$  by  $I|_{\tau'}$ . For a  $\tau$ -interpretation  $I$ , a truth value  $v$  and an atom  $P \in \tau$ , we denote by  $I[P/v]$  the  $\tau$ -interpretation that assigns  $v$  to  $P$  and corresponds to  $I$  for all other atoms. We extend this notation to sets of atoms. Both truth and precision order can be extended to an order on all  $\tau$ -interpretations by  $I \leq J$  if for each atom  $P \in \tau$ ,  $I(P) \leq J(P)$  and  $I \leq_p J$  if for each atom  $P \in \tau$ ,  $I(P) \leq_p J(P)$ .

A three-valued interpretation  $I$  on  $\tau$  can be extended to all PC-formulas over  $\tau$  by induction on the subformula order:

- $P^I = I(P)$  if  $P \in \tau$ ;
- $(\varphi \wedge \psi)^I = \min_{\leq}(\{\varphi^I, \psi^I\})$ ;
- $(\varphi \vee \psi)^I = \max_{\leq}(\{\varphi^I, \psi^I\})$ ;
- $(\neg\varphi)^I = (\varphi^I)^{-1}$ .

The following proposition states a well-known monotonicity property with respect to the precision order.

**Proposition 1.** *Let  $\varphi$  be a PC-formula over  $\tau$  and  $I, J$  be three-valued  $\tau$ -interpretations such that  $I \leq_p J$ . Then  $\varphi^I \leq_p \varphi^J$ .*

Another well-known proposition states a monotonicity property with respect to the truth order.

**Proposition 2.** *Let  $\varphi$  be a PC-formula over  $\tau$  and  $I, J$  be three-valued  $\tau$ -interpretations such that if  $P^I < P^J$ , then  $P$  only occurs positively in  $\varphi$  and if  $P^I > P^J$  then  $P$  only occurs negatively in  $\varphi$ . Then  $\varphi^I \leq \varphi^J$ .*

The above properties about the precision and truth order will be applied frequently in the proofs in Section 4. For brevity, we will not mention them explicitly in the remainder of the paper.

We now define the semantics of definitions. Let  $D$  be a definition over  $\tau$  and  $I_O$  a two-valued  $\tau_D^o$ -interpretation, i.e., an interpretation of all open symbols of  $D$ . Consider a sequence of three-valued  $\tau$ -interpretations  $(I^n)_{n \geq 0}$  extending  $I_O$  such that  $I^0(P) = \mathbf{u}$  for every  $P \in \tau_D^d$ , and for every natural number  $n$ ,  $I^{n+1}$  relates to  $I^n$  in one of the following ways:

1.  $I^{n+1} = I^n[P/\mathbf{t}]$  where  $P$  is a defined atom such that  $P^{I^n} = \mathbf{u}$  and for some rule  $P \leftarrow \varphi \in D$ ,  $\varphi^{I^n} = \mathbf{t}$ .
2.  $I^{n+1} = I^n[U/\mathbf{f}]$ , where  $U$  is a non-empty set of defined atoms, such that for each  $P \in U$ ,  $I^n(P) = \mathbf{u}$  and for each rule  $P \leftarrow \varphi \in D$ ,  $\varphi^{I^{n+1}} = \mathbf{f}$ .

The first derivation rule 1 derives true atoms and is a straightforward formalization of the principle explained in the beginning of this section. The second derivation rule 2 is less obvious and serves to derive falsity of defined atoms. Let us first consider a more obvious special case that is subsumed by rule 2:

3.  $I^{n+1} = I^n[P/\mathbf{f}]$  where  $P$  is a defined atom such that  $I^n(P) = \mathbf{u}$  and for each rule  $P \leftarrow \varphi \in D$ ,  $\varphi^{I^n} = \mathbf{f}$ .

This rule expresses that if the body of each rule that could derive  $P$  is certainly false at stage  $n$ , then  $P$  can be asserted to be false at stage  $n + 1$ . This is a special case of the rule 2. Indeed, taking  $U = \{P\}$ , we have for each  $P \leftarrow \varphi \in D$  that  $\mathbf{f} = \varphi^{I^n} \leq_p \varphi^{I^n[U/\mathbf{f}]} = \varphi^{I^{n+1}} = \mathbf{f}$ .

The stronger derivation rule 2 expresses that the atoms in a set  $U$  consisting of undervived defined atoms can be turned to false if the assumption that they are all false suffices to dissatisfy the condition of each rule that could produce an element of  $U$ . A set  $U$  as used in this rule corresponds exactly to an *unfounded set* as defined in [20]. The rationale behind this derivation rule and the link with informal induction is that when  $U$  is an unfounded set at stage  $n$  then none of its atoms can be derived anymore at later stages of the construction process (using derivation rule 1). To see this, assume towards contradiction that at some later stage  $> n$ , one or more elements of  $U$  could be derived to be true, and let  $P$  be the first atom that could be derived, say at stage  $m > n$ . At stage  $m$ , it holds for each  $Q \in U$  that  $I^m(Q) = \mathbf{u}$  and for some rule  $P \leftarrow \varphi \in D$ ,  $\varphi^{I^m} = \mathbf{t}$ . But  $I^n[U/\mathbf{f}] \leq_p I^m[U/\mathbf{f}] \geq_p I^m$  and hence,  $\mathbf{f} = \varphi^{I^n[U/\mathbf{f}]} \leq_p \varphi^{I^m[U/\mathbf{f}]} \geq_p \varphi^{I^m} = \mathbf{t}$  and this yields a contradiction. Thus, the derivation rule 2 correctly concludes that the atoms in  $U$  are no longer derivable through rule application. This derivation rule is needed to derive, e.g., falsity of all atoms not in the least fixpoint of a monotone definition, which is something that cannot be derived in general by the rule 3.

We call a sequence as defined above a *well-founded induction*. A well-founded induction is *terminal* if it cannot be extended anymore. It can be shown that each terminal well-founded induction is a sequence of increasing precision and its limit is the *well-founded partial interpretation* of  $D$  extending  $I_O$  [15]. We denote the well-founded partial interpretation of  $D$  extending  $I_O$  by  $I_O^D$ .

We define that  $D^I = \mathbf{t}$  if  $I = (I|_{\tau_D^D})^D$  and  $I$  is two-valued. Otherwise, we define  $D^I = \mathbf{f}$ . Adding this as a new base case to the definition of the truth function of formulas, we can extend the truth function inductively to all PC(ID)-formulas.

We are now ready to define the semantics of PC(ID). For an arbitrary PC(ID)-formula  $\varphi$ , we say that an interpretation  $I$  satisfies  $\varphi$ , or  $I$  is a model of  $\varphi$ , if  $I$  is two-valued and  $\varphi^I = \mathbf{t}$ . As usual, this is denoted by  $I \models \varphi$ .  $I$  satisfies (is a model of) a PC(ID) theory  $T$  if  $I$  satisfies every  $\varphi \in T$ .

A definition lays a functional relation between the interpretation of the defined symbols and those of the open symbols. In particular, two models of a definition differ on the open symbols. A model of a monotone definition is the  $\leq$ -least interpretation satisfying the rules of the definition (interpreted as material implications) given a fixed interpretation of the open symbols, as desired. Also, the semantics of PC(ID) is two-valued and extends the standard semantics of propositional logic. A three-valued interpretation  $I$  is never a model of a definition, not even if it is a well-founded partial interpretation of the definition.

*Example 2.* Consider the following definition:

$$D = \left\{ \begin{array}{l} P \leftarrow Q \\ Q \leftarrow P \end{array} \right\}.$$

Then  $\tau_D^o = \emptyset$  and  $\tau_D^d = \{P, Q\}$ . There are no open symbols and there is only one model of  $D$ , namely the interpretation mapping both  $P$  and  $Q$  to **f**.

### 2.3 Where the informal semantics breaks

The informal semantics of a PC(ID) rule set as an inductive definition breaks in some cases. Examples are non-monotone rule sets with recursion over negation such as

$$\{ P \leftarrow \neg P \}$$

or

$$\left\{ \begin{array}{l} P \leftarrow \neg Q \\ Q \leftarrow \neg P \end{array} \right\}$$

Their (unique) well-founded partial interpretation is not two-valued, and hence, these definitions have no model and are inconsistent in PC(ID).

The restriction to two-valued well-founded partial models was imposed to enforce the view that a well-designed definition  $D$  ought to define the truth of all its defined atoms, i.e., the inductive process should be able to derive truth or falsity of all defined atoms. This motivates the following concept.

**Definition 3 (Totality, [14]).** *Let  $I_O$  be a two-valued interpretation of  $\tau_D^o$ . A definition  $D$  is total in  $I$  if  $I_O^D$  is two-valued. The definition  $D$  is total in the context of a theory  $T$  if  $D$  is total in  $M|_{\tau_D^o}$ , for each model  $M$  of  $T$ . A definition  $D$  is total if it is total in every two-valued interpretation  $I_O$  of its open atoms.*

A simple and very general syntactic criterion that guarantees that a definition is total can be phrased in terms of the dependency relation  $\prec$  of  $D$ . A definition  $D$  is *stratified* if for each rule  $P \leftarrow \varphi$ , for each symbol  $Q$  with a negative occurrence in  $\varphi$ ,  $P \not\prec Q$ . This means that the definition of  $Q$  does not depend on  $P$ .

**Proposition 3 ([20]).** *If  $D$  is stratified then  $D$  is total.*

Observe that a stratified definition formally satisfies the (informal) condition that was stated for iterated inductive definitions early in this section. The well-founded semi-order underlying an iterated inductive definition is nothing else than the reflexive closure  $\preceq$  of  $\prec$ . Atoms  $Q$  with a positive occurrence in the body of a rule deriving  $P$  satisfy  $Q \preceq P$ ; those with a negative occurrence satisfy  $Q \preceq P$  and  $P \not\preceq Q$ . Hence, such rules effectively derive  $P$  given the presence of less or equivalent atoms and the absence of strictly less atoms in the defined valuation. The well-founded model of such definitions is two-valued and corresponds exactly to the structure obtained by the construction described in Section 2.2 for (informal) inductive definitions. Thus, the well-founded semantics correctly formalizes the informal semantics of inductive definitions, and correctly constructs the (informally) defined relations without knowing the underlying (semi-)order of the definition.

Although the class of stratified definitions is large and comprises almost all “practical” PC(ID) definitions that we encountered in applications, there are intuitively sensible definitions which are total but not stratified.

*Example 3.* A software system consists of two servers  $S1$  and  $S2$  that provide identical services. One server acts as master and the other as slave, and these roles are assigned on the basis of clear (but irrelevant) criterion that can be expressed in the form of a set of defining rules for the predicate  $Master(s)$ . Clients can request services  $x$ . The master makes a selection among these requests on the basis of a clear (but irrelevant) criterion expressed in a definition of  $Criterion(x)$ . The slave fulfills all requests that are not accepted by the master. Here is the core of a (predicate) definition:

$$\left\{ \begin{array}{l} Criterion(x) \leftarrow \dots \\ Master(s) \leftarrow \dots \\ Slave(s) \leftarrow \neg Master(s) \\ Accepts(x, m) \leftarrow Request(x) \wedge Master(m) \wedge Criterion(x) \\ Accepts(x, s) \leftarrow Request(x) \wedge Slave(s) \wedge \exists m (Master(m) \wedge \neg Accepts(x, m)) \end{array} \right\}$$

The (propositionalisation of the) definition is not stratified since the last rule creates a negative dependency between  $Accepts(x, S1)$  and  $Accepts(x, S2)$ . Yet, since no server can be both master and slave, this recursion is broken “locally” in each model. This is a total, albeit unstratified definition of the predicate  $Accepts$  that correctly implements the informal specification.

The proof system for PC(ID), as presented below, is sound and complete with respect to all PC(ID) theories containing only total definitions, and hence to any fragment of PC(ID) that enforces totality of the allowed definitions.

### 3 LPC(ID): A Proof system for PC(ID)

In this section we formulate a proof system, **LPC(ID)**, for the logic PC(ID) in the sequent calculus style originally developed by Gentzen in 1935 [21]. Our system can be seen essentially as a propositional part of classical sequent calculus adaptation of inference rules for definitions. We give the proof rules of **LPC(ID)**, which are the rules of Gentzen’s original sequent calculus for propositional logic, augmented with rules for introducing defined atoms on the left and right of sequents, a rule for inferring the non-totality of definitions and a rule for introducing definitions on the right of sequents.

First, we introduce some basic definitions and notations. Let capital Greek letters  $\Gamma, \Delta, \dots$  denote finite (possibly empty) sets of PC(ID)-formulas.  $\Gamma, \Delta$  denotes  $\Gamma \cup \Delta$ .  $\Gamma, \varphi$  denotes  $\Gamma \cup \{\varphi\}$ . By  $\bigwedge \Gamma$ , respectively  $\bigvee \Gamma$ , we denote the conjunction, respectively disjunction of all formulas in  $\Gamma$ . By  $\neg \Gamma$ , we denote the set obtained by taking the negation of each formula in  $\Gamma$ . By  $\Gamma \setminus \Delta$ , we denote the set obtained by deleting from  $\Gamma$  all occurrences of formulas that occur in  $\Delta$ .  $\Gamma$  is said to be *consistent* if there is no formula  $\varphi$  such that both  $\varphi$  and  $\neg \varphi$  can be derived from  $\Gamma$ .

A *sequent* is an expression of the form  $\Gamma \rightarrow \Delta$ .  $\Gamma$  and  $\Delta$  are respectively called the *antecedent* and *succedent* of the sequent and each formula in  $\Gamma$  and  $\Delta$  is called a *sequent formula*. In general, a formula  $\varphi$  occurring as part of a sequent denotes the set  $\{\varphi\}$ . We will denote sequents by  $S, S_1, \dots$ . A sequent  $\Gamma \rightarrow \Delta$  is *valid*, denoted

by  $\models \Gamma \rightarrow \Delta$ , if every model of  $\bigwedge \Gamma$  satisfies  $\bigvee \Delta$ . A *counter-model* for  $\Gamma \rightarrow \Delta$  is an interpretation  $I$  such that  $I \models \bigwedge \Gamma$  but  $I \not\models \bigvee \Delta$ . The sequent  $\Gamma \rightarrow$  is equivalent to  $\Gamma \rightarrow \perp$  and  $\rightarrow \Delta$  is equivalent to  $\top \rightarrow \Delta$ , where  $\perp, \top$  are logical constants denoting *false* and *true*, respectively.

An *inference rule* is an expression of the form

$$\frac{S_1; \dots; S_n}{S} \quad n \geq 0$$

where  $S_1, \dots, S_n$  and  $S$  are sequents. Each  $S_i$  is called a *premise* of the inference rule,  $S$  is called the *consequence*. Intuitively, an inference rule means that  $S$  can be inferred, given that all  $S_1, \dots, S_n$  are already inferred.

The *initial sequents*, or *axioms* of **LPC(ID)** are all sequents of the form

$$\Gamma, A \rightarrow A, \Delta \quad \text{or} \quad \perp \rightarrow \Delta \quad \text{or} \quad \Gamma \rightarrow \top$$

where  $A$  is any PC(ID)-formula,  $\Gamma$  and  $\Delta$  are arbitrary sets of PC(ID)-formulas.

The inference rules for **LPC(ID)** consist of *structural* rules, *logical* rules and *definition* rules. The structural and logical rules, which follow directly the propositional inference rules in Gentzen's original sequent calculus for first-order logic **LK**, deal with the propositional part of PC(ID) and are given as follows, in which  $A, B$  are any PC(ID)-formulas and  $\Gamma, \Delta$  are arbitrary sets of PC(ID)-formulas.

### Structural rules

- Weakening rules

$$\text{left: } \frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}; \quad \text{right: } \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}.$$

- Contraction rules

$$\text{left: } \frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}; \quad \text{right: } \frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}.$$

- Cut rule

$$\frac{\Gamma \rightarrow \Delta, A; \quad A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}.$$

### Logical rules

- $\neg$  rules

$$\text{left: } \frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}; \quad \text{right: } \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}.$$

- $\wedge$  rules

$$\text{left: } \frac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}; \quad \text{right: } \frac{\Gamma \rightarrow \Delta, A; \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}.$$

- $\vee$  rules

$$\text{left: } \frac{A, \Gamma \rightarrow \Delta; \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}; \quad \text{right: } \frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B}.$$

Our deductive system **LPC(ID)** is then obtained from the propositional part of **LK** by adding inference rules for definitions. The definition rules of **LPC(ID)** consist of the *right definition* rule, the *left definition* rule, the *non-total* definition rule and the *definition introduction* rule. Without loss of generality, in what follows we assume that there is only one rule with head  $P$  in a definition  $D$  for every  $P \in \tau_D^d$ . We refer to this rule as *the rule for  $P$  in  $D$*  and denote it by  $P \leftarrow \varphi_P$ .

**Right definition rule for  $P$ .** The *right definition rule* introduces defined atoms in the succedents of sequents. It allows inferring the truth of a defined atom from a definition  $D$  and is therefore closely related to the derivation rule 1 for extending a well-founded induction. Let  $D$  be a definition and  $P$  a defined atom of  $D$ . The right definition rule for  $P$  is given as follows.

$$\frac{\Gamma \rightarrow \Delta, \varphi_P}{D, \Gamma \rightarrow \Delta, P}$$

where  $\Gamma$  and  $\Delta$  are arbitrary sets of PC(ID)-formulas.

We illustrate this inference rule with an example.

*Example 4.* Consider the definition

$$D = \left\{ \begin{array}{l} P \leftarrow P \wedge \neg Q \\ Q \leftarrow \neg P \end{array} \right\}.$$

The instance of the right definition rule for  $P$  is

$$\frac{\Gamma \rightarrow \Delta, P \wedge \neg Q}{D, \Gamma \rightarrow \Delta, P},$$

and the instance of the right definition rule for  $Q$  is

$$\frac{\Gamma \rightarrow \Delta, \neg P}{D, \Gamma \rightarrow \Delta, Q}.$$

**Left definition rule for  $P_i \in U$ .** The *left definition rule* introduces defined atoms in the antecedents of sequents. It allows inferring the falsity of a defined atom from a definition  $D$  and is therefore closely related to the second derivation rule 2 for extending a well-founded induction.

We first introduce some notations. Given a set  $U$  of atoms, let  $U^\triangleright$  be a set consisting of one new atom  $P^\triangleright$  for every  $P \in U$ . The vocabulary  $\tau$  augmented with these symbols is denoted by  $\tau^\triangleright$ . Given a PC-formula  $\varphi$ ,  $\varphi^\triangleright$  denotes the formula obtained by replacing all positive occurrences of an atom  $P \in U$  in  $\varphi$  by  $P^\triangleright$ . We call  $\varphi^\triangleright$  the *renaming* of  $\varphi$  with respect to  $U$ . For a set of PC-formulas  $F$ ,  $F^\triangleright$  denotes  $\{\varphi^\triangleright \mid \varphi \in F\}$ . For arbitrary PC-formula  $\varphi$ , by  $\neg\varphi^\triangleright$ , we mean  $\neg(\varphi^\triangleright)$ .

Let  $D$  be a definition over  $\tau$  and  $U$  a non-empty set of atoms such that  $U \subseteq \tau_D^d$ . Denote by  $\neg U^\triangleright$  the set  $\{\neg P^\triangleright \mid P \in U\}$ . Let  $\Gamma$  and  $\Delta$  be sets of PC(ID)-formulas over  $\tau$ . The left definition rule for every  $P_i \in U$  is given as follows, where  $U = \{P_1, \dots, P_n\}$ .

$$\frac{\neg U^\triangleright, \Gamma \rightarrow \Delta, \neg \varphi_{P_1}^\triangleright; \dots; \neg U^\triangleright, \Gamma \rightarrow \Delta, \neg \varphi_{P_n}^\triangleright}{P_i, D, \Gamma \rightarrow \Delta}.$$

Actually, in the left definition rule, the set of atoms  $U$  is a candidate unfounded set of  $D$ .

We illustrate this inference rule with an example.

*Example 5.* Given a definition  $D = \left\{ \begin{array}{l} P \leftarrow P \wedge \neg Q \\ Q \leftarrow Q \end{array} \right\}$ ,

–  $U = \{P\}$ , the instance of the left definition rule for  $P \in U$  is

$$\frac{\neg P^\triangleright, \Gamma \rightarrow \Delta, \neg(P^\triangleright \wedge \neg Q)}{P, D, \Gamma \rightarrow \Delta}$$

–  $U = \{Q\}$ , the instance of the left definition rule for  $Q \in U$  is

$$\frac{\neg Q^\triangleright, \Gamma \rightarrow \Delta, \neg Q^\triangleright}{Q, D, \Gamma \rightarrow \Delta}$$

–  $U = \{P, Q\}$ , the instance of the left definition rule for  $P \in U$  is

$$\frac{\neg P^\triangleright, \neg Q^\triangleright, \Gamma \rightarrow \Delta, \neg(P^\triangleright \wedge \neg Q); \quad \neg P^\triangleright, \neg Q^\triangleright, \Gamma \rightarrow \Delta, \neg Q^\triangleright}{P, D, \Gamma \rightarrow \Delta}$$

–  $U = \{P, Q\}$ , the instance of the left definition rule for  $Q \in U$  is

$$\frac{\neg P^\triangleright, \neg Q^\triangleright, \Gamma \rightarrow \Delta, \neg(P^\triangleright \wedge \neg Q); \quad \neg P^\triangleright, \neg Q^\triangleright, \Gamma \rightarrow \Delta, \neg Q^\triangleright}{Q, D, \Gamma \rightarrow \Delta}.$$

**Non-total definition rule for  $D$ .** The *non-total definition rule* allows inferring the non-totality of a definition  $D$ . We introduce some notations. Let  $D$  be a definition over  $\tau$  and  $V$  a non-empty set of atoms such that  $V \subseteq \tau_D^d$ . Denote by  $\tau^\diamond$  the vocabulary  $\tau \cup V^\triangleright \cup V^\diamond$ , where both  $V^\triangleright$  and  $V^\diamond$  are sets of new and different renamings  $P^\triangleright$  and  $P^\diamond$  of all symbols  $P$  of  $V$ . Denote by  $\varphi^\diamond$  the formula obtained by replacing each positive occurrence of each  $P \in V$  in  $\varphi$  by  $P^\triangleright$  and each negative occurrence of each  $P \in V$  in  $\varphi$  by  $P^\diamond$ . Denote by  $D^\diamond$  the definition  $\{P^\triangleright \leftarrow \varphi_P^\diamond \mid P \in V \text{ and } P \leftarrow \varphi_P \in D\}$  over the new vocabulary  $\tau^\diamond$ . Let  $\Gamma$  and  $\Delta$  be sets of PC(ID)-formulas over  $\tau$ . Then the non-total definition rule for  $D$  is given as follows.

$$\frac{V^\diamond, D^\diamond, \Gamma \rightarrow \Delta, \bigwedge \neg V^\triangleright; \quad \neg V^\diamond, D^\diamond, \Gamma \rightarrow \Delta, \bigwedge V^\triangleright}{D, \Gamma \rightarrow \Delta}$$

We illustrate this inference rule with an example.

*Example 6.* Given a definition  $D = \left\{ \begin{array}{l} P \leftarrow P \wedge \neg Q \\ Q \leftarrow \neg Q \wedge R \\ R \leftarrow \neg R \end{array} \right\}$ ,  $V = \{Q, R\}$  and  $\Gamma$  and  $\Delta$  empty sets. Then the instance of the non-total definition for  $D$  is

$$\frac{Q^\diamond, R^\diamond, D^\diamond \rightarrow \neg Q^\triangleright \wedge \neg R^\triangleright; \neg Q^\diamond, \neg R^\diamond, D^\diamond \rightarrow Q^\triangleright \wedge R^\triangleright}{D \rightarrow},$$

where  $D^\diamond = \left\{ \begin{array}{l} Q^\triangleright \leftarrow \neg Q^\diamond \wedge R^\triangleright \\ R^\triangleright \leftarrow \neg R^\diamond \end{array} \right\}$ .

For the intuition behind the non-total definition rule, we point the readers to [14] and Section 2.3 where the cause of the non-totality of a definition is explained.

We do not have an inference rule to prove totality of all definitions in the context of a certain set  $\Gamma$  of PC(ID)-formulas. Such an inference rule would involve proving that each model of  $\Gamma$  can be extended to a model of the definition. In fact, we cannot even formulate this condition as a sequent.

**Definition introduction rule for  $D$ .** The three definitional inference rules introduced so far, introduce a definition in the antecedent of the consequence. Hence, none of these rules can be used to infer that under certain conditions a definition holds. The *definition introduction rule* allows inferring the truth of a total definition from PC(ID)-formulas.

We introduce some notations. Let  $D$  be a total definition. Denote by  $P'$  a new defined atom for each  $P \in \tau_D^d$ . Denote by  $\tau'$  the vocabulary  $\tau \cup \{P' \mid P \in \tau_D^d\}$ . Denote by  $D'$  the definition over the new vocabulary  $\tau'$  obtained by replacing each occurrence of each defined symbol  $P$  in  $D$  by  $P'$ . Let  $\Gamma$  and  $\Delta$  be sets of PC(ID)-formulas over the old vocabulary  $\tau$ . The definition introduction rule for  $D$  is given as follows, where  $P_1, \dots, P_n$  are all defined atoms of  $D$ .

$$\frac{D', \Gamma \rightarrow \Delta, P'_1 \equiv P_1; \dots; D', \Gamma \rightarrow \Delta, P'_n \equiv P_n}{\Gamma \rightarrow \Delta, D}$$

We illustrate this inference rule with an example.

*Example 7.* Given a definition  $D = \left\{ \begin{array}{l} P \leftarrow O \\ Q \leftarrow Q \wedge P \end{array} \right\}$ ,  $\Gamma = \{O, P, \neg Q\}$  and  $\Delta$  an empty set. Then the instance of the definition introduction rule for  $D$  is

$$\frac{D', O, P, \neg Q \rightarrow P' \equiv P; \quad D', O, P, \neg Q \rightarrow Q' \equiv Q}{O, P, \neg Q \rightarrow D},$$

where  $D' = \left\{ \begin{array}{l} P' \leftarrow O \\ Q' \leftarrow Q' \wedge P' \end{array} \right\}$ .

The inference rule proposed here has a definition in the succedent of its premise and hence, allows to infer the truth of a definition. Unfortunately, this rule is only sound given that the inferred definition is total. We will give an example to show that the definition introduction rule is not sound given that the inferred definition is non-total right after proving the soundness of this inference rule.



**Proofs of PC(ID).** We now come to the notion of an **LPC(ID)**-proof for a sequent.

**Definition 4.** An **LPC(ID)**-proof for a sequent  $S$ , is a tree  $T$  of sequents with root  $S$ . Moreover, each leaf of  $T$  must be an axiom and for each interior node  $S'$  there exists an instance of an inference rule such that  $S'$  is the consequence of that instance while the children of  $S'$  are precisely the premises of that instance.  $T$  is often called a proof tree for  $S$ . A sequent  $S$  is called provable in **LPC(ID)**, or **LPC(ID)**-provable, if there is an **LPC(ID)**-proof for it.

*Example 8.* Given a definition  $D = \left\{ \begin{array}{l} P \leftarrow O \\ Q \leftarrow Q \wedge P \end{array} \right\}$ , the following is an **LPC(ID)**-proof for  $O, D \rightarrow P \wedge \neg Q$ .

$$\begin{array}{c}
 \frac{Q^\triangleright \rightarrow Q^\triangleright}{\neg Q^\triangleright, Q^\triangleright \rightarrow} \text{left } \neg \\
 \frac{\neg Q^\triangleright, Q^\triangleright \rightarrow}{\neg Q^\triangleright, Q^\triangleright, P \rightarrow} \text{left weakening} \\
 \frac{\neg Q^\triangleright, Q^\triangleright, P \rightarrow}{\neg Q^\triangleright, Q^\triangleright \wedge P \rightarrow} \text{left } \wedge \\
 \frac{\neg Q^\triangleright, Q^\triangleright \wedge P \rightarrow}{\neg Q^\triangleright \rightarrow \neg(Q^\triangleright \wedge P)} \text{right } \neg \\
 \frac{\neg Q^\triangleright \rightarrow \neg(Q^\triangleright \wedge P)}{Q, D \rightarrow} \text{left definition rule} \\
 \frac{Q, D \rightarrow}{D \rightarrow \neg Q} \text{right } \neg \\
 \frac{D \rightarrow \neg Q}{O, D \rightarrow \neg Q} \text{left weakening} \\
 \frac{O \rightarrow O}{O, D \rightarrow P} \text{right definition rule} \\
 \frac{O, D \rightarrow P \quad O, D \rightarrow \neg Q}{O, D \rightarrow P \wedge \neg Q} \text{right } \wedge
 \end{array}$$

## 4 Main results

In this section, we will prove that the deductive system **LPC(ID)** is sound and complete for a slightly restricted fragment of PC(ID), which can be viewed as main theoretical results of this paper.

### 4.1 Soundness

To prove the soundness of **LPC(ID)**, it is sufficient to prove that all axioms of **LPC(ID)** are valid and that every inference rule of **LPC(ID)** is sound, i.e. if all premises of an inference rule are valid then the consequence of that rule is valid. It is trivial to verify that the axioms are valid and that the structural and logical rules are sound (see e.g. [36,37]). Hence, only the soundness of the right definition rule, the left definition rule, the non-total definition rule and the definition introduction rule must be proved.

**Lemma 1.** *Let  $I$  be a model of  $D$  and  $P$  a defined atom of  $D$ . Then  $I \models P$  if and only if  $I \models \varphi_P$ .*

*Proof.* Because  $I$  is a model of  $D$ , there exists a terminal well-founded induction  $(I^n)_{n \leq \xi}$  for  $D$  with the limit  $I^\xi = I$ .

(if part) Assume that  $I \models \varphi_P$ . The sequence  $(I^n)_{n \leq \xi}$  is strictly increasing in precision, hence there is no  $n \leq \xi$  such that  $\varphi_P^{I^n} = \mathbf{f}$ . As such, for every  $n \leq \xi$ ,  $P^{I^n} \neq \mathbf{f}$ . Therefore,  $P^I \neq \mathbf{f}$  and because  $I$  is two-valued, we can conclude  $P^I = \mathbf{t}$ .

(only if part) Assume that  $I \models P$ . Thus, for some  $n < \xi$ ,  $P^{I^n} = \mathbf{u}$  and  $P^{I^{n+1}} = \mathbf{t}$ . Hence,  $\varphi_P^{I^n} = \mathbf{t}$ . Because the sequence  $(I^n)_{n \leq \xi}$  is strictly increasing in precision, we have  $\varphi_P^I = \mathbf{t}$ .

**Lemma 2 (Soundness of the right definition rule).** *Let  $D$  be a definition and  $P$  a defined atom of  $D$ . If  $\models \Gamma \rightarrow \Delta, \varphi_P$ , then  $\models D, \Gamma \rightarrow \Delta, P$ .*

*Proof.* Assume  $\models \Gamma \rightarrow \Delta, \varphi_P$  but  $\not\models D, \Gamma \rightarrow \Delta, P$ . Then there exists a counter-model  $I$  for  $D, \Gamma \rightarrow \Delta, P$  which satisfies  $D, \bigwedge \Gamma, \neg \bigvee \Delta$  and  $\neg P$ . It follows from the first assumption that  $I \models \varphi_P$ , and hence, by Lemma 1,  $I \models P$ , a contradiction.

**Lemma 3 (Soundness of the left definition rule).** *Let  $D$  be a definition and  $U$  be a non-empty subset of  $\tau_D^d$ . If for every  $P \in U$ , it holds that  $\models \neg U^\triangleright, \Gamma \rightarrow \Delta, \neg \varphi_P^\triangleright$ , then for all  $P \in U$ , it holds that  $\models P, D, \Gamma \rightarrow \Delta$ .*

*Proof.* Assume  $\models \neg U^\triangleright, \Gamma \rightarrow \Delta, \neg \varphi_P^\triangleright$  for every  $P \in U$ , but  $\not\models P, D, \Gamma \rightarrow \Delta$  for some  $P \in U$ . Then there exists a model  $I$  of  $D, \bigwedge \Gamma$  and  $\neg \bigvee \Delta$  satisfying at least one  $P \in U$ . Furthermore, by Lemma 1, it holds that  $I \models \varphi_P$ . We select this  $P$  in the following way. Let  $(I^n)_{n \leq \xi}$  be a terminal well-founded induction for  $D$  with limit  $I^\xi = I$ . Let  $n$  be the smallest  $n \leq \xi$  such that for some  $Q \in U$ ,  $Q^{I^n} = \mathbf{u}$  and  $Q^{I^{n+1}} = \mathbf{t}$ . By selection of  $n$ , there is a unique  $P \in U$  such that  $P^{I^n} = \mathbf{u}$ ,  $I^n \models \varphi_P$  and  $P^{I^{n+1}} = \mathbf{t}$ . Consider this  $P$  and  $\varphi_P$ .

On the one hand, it holds that  $I \models \varphi_P$ . On the other hand, consider the interpretation  $I^\triangleright = I[U^\triangleright/\mathbf{f}]$ . It is clear that  $I^\triangleright$  satisfies  $\neg U^\triangleright, \bigwedge \Gamma$  and  $\neg \bigvee \Delta$ . Hence, by the first assumption, it holds that  $I^\triangleright \models \neg \varphi_P^\triangleright$ . We will derive a contradiction from this.

Observe that by our choice of  $n$ , for each  $Q \in U$ ,  $Q^{I^n} = \mathbf{f}$  or  $Q^{I^n} = \mathbf{u}$ . Denote by  $I^{n^\triangleright}$  the interpretation that assigns  $Q^{I^n}$  to  $Q^\triangleright$  for every  $Q \in U$  and corresponds to  $I^n$  on all other atoms. There are two simple observations that can be made about  $I^{n^\triangleright}$ :

- $I^{n^\triangleright} \leq_p I^\triangleright$ : indeed,  $I^n \leq_p I$  and for each  $Q^\triangleright \in U^\triangleright$ ,  $Q^\triangleright^{I^\triangleright} = \mathbf{f} \geq_p Q^\triangleright^{I^{n^\triangleright}} = Q^{I^n} = \mathbf{f}$  or  $\mathbf{u}$ .
- $(\varphi_P^\triangleright)^{I^{n^\triangleright}} = \varphi_P^{I^n} = \mathbf{t}$ : obvious from the construction of  $I^{n^\triangleright}$  and  $\varphi_P^\triangleright$ .

Combining these results, we obtain  $\mathbf{t} = (\varphi_P^\triangleright)^{I^{n^\triangleright}} \leq_p (\varphi_P^\triangleright)^{I^\triangleright} = \mathbf{f}$ . This is the desired contradiction.

Having the soundness of the left definition rule, we can explain the introduction of renaming formulas in the left definition rule. Consider the left definition rule of the following form:

$$\frac{\neg P_1, \dots, \neg P_n, \Gamma \rightarrow \Delta, \neg \varphi_{P_1}; \dots; \neg P_1, \dots, \neg P_n, \Gamma \rightarrow \Delta, \neg \varphi_{P_n}}{P_i, D, \Gamma \rightarrow \Delta} \quad (1)$$

where  $\{P_1, \dots, P_n\} \subseteq \tau_D^d$  and  $P_i$  is an arbitrary defined atom in  $\{P_1, \dots, P_n\}$ .

Intuitively, the above form of the left definition rule is exactly related to the second derivation rule 2 of the well-founded induction and it is easier to be understood. However, such an inference rule is not sound. For an arbitrary definition  $D$  and any defined atom  $P$  of  $D$ ,  $D \rightarrow \neg P$  can be inferred applying this rule. We illustrate this with the next example.

*Example 9.* Consider the following definition:

$$D = \{ P \leftarrow \top \}.$$

Let  $\Gamma = \{P\}$  and  $\Delta$  be an empty set. Since  $\neg P, P \rightarrow \neg \top$ , we can prove  $D \rightarrow \neg P$  by using the inference rule (1), the right  $\neg$  rule and the right contraction rule. However, for the same definition  $D$  and empty sets  $\Gamma$  and  $\Delta$ , it is obvious that  $D \rightarrow P$  can be inferred by using the right definition rule, which derives a contradiction. Hence, the inference rule (1) is not sound.

From the viewpoint of semantics, since the left definition rule corresponds to the second derivation rule 2 of the well-founded induction, we have to adopt the approach of renaming to represent that the defined atoms of  $U$  are unknown in  $I^n$  and false in  $I^{n+1}$ .

**Lemma 4.** *Let  $D$  be a definition,  $I$  a model of  $D$  and  $U$  a non-empty subset of  $\tau_D^d$ . If for every  $P \in U$ , it holds that  $\varphi_P^{I[U/\mathbf{f}]} = \mathbf{f}$ , then  $P^I = \mathbf{f}$  for all  $P \in U$ .*

*Proof.* Assume that there exists a non-empty set  $T$  satisfying that (a)  $T \subseteq U$ , (b)  $P^I = \mathbf{t}$  for each  $P \in T$ , and (c)  $P^I = \mathbf{f}$  for each  $P \in U \setminus T$ . Let  $(I^n)_{n \leq \xi}$  be a terminal well-founded induction for  $D$  with the limit  $I^\xi = I$ . Let  $n$  be the smallest  $n \leq \xi$  such that for some  $Q \in T$ ,  $Q^{I^n} = \mathbf{u}$  and  $Q^{I^{n+1}} = \mathbf{t}$ . By selection of  $n$ , there is a unique  $P \in T$  such that  $P^{I^n} = \mathbf{u}$ ,  $\varphi_P^{I^n} = \mathbf{t}$  and  $P^{I^{n+1}} = \mathbf{t}$ . Consider this  $P$  and  $\varphi_P$ .

Observe that by our choice of  $n$ , for each  $Q \in T$ ,  $Q^{I^n} = \mathbf{u}$ . Hence, for each  $Q \in T$ ,  $Q^{I^n} \leq_p Q^{I[U/\mathbf{f}]} = \mathbf{f}$ . Because  $I^n \leq_p I$ , for each  $Q \in \tau_D^d \setminus T$ , we have that  $Q^{I[U/\mathbf{f}]} = Q^I \geq_p Q^{I^n}$ . Combining these results, it is concluded that  $I^n \leq_p I[U/\mathbf{f}]$ . Therefore, we obtain that  $\mathbf{t} = \varphi_P^{I^n} \leq_p \varphi_P^{I[U/\mathbf{f}]} = \mathbf{f}$ , a contradiction. Hence, there is no  $P \in U$  such that  $P^I = \mathbf{t}$ , which follows directly that  $P^I = \mathbf{f}$  for all  $P \in U$ .

**Lemma 5 (Soundness of the non-total definition rule).** *If  $\models V^\diamond, D^\diamond, \Gamma \rightarrow \Delta, \bigwedge \neg V^\triangleright$  and  $\models \neg V^\diamond, D^\diamond, \Gamma \rightarrow \Delta, \bigwedge V^\triangleright$ , then  $\models D, \Gamma \rightarrow \Delta$ .*

*Proof.* Assume towards contradiction that

$$\models V^\diamond, D^\diamond, \Gamma \rightarrow \Delta, \bigwedge \neg V^\triangleright \text{ and } \models \neg V^\diamond, D^\diamond, \Gamma \rightarrow \Delta, \bigwedge V^\triangleright \text{ but } \not\models D, \Gamma \rightarrow \Delta. \quad (2)$$

Then there exists a  $\tau$ -interpretation  $I$  satisfying  $D, \bigwedge \Gamma$  and  $\neg \bigvee \Delta$ . Consider the vocabulary  $\tau^\diamond = \tau \cup V^\triangleright \cup V^\diamond$ .  $I$  can be expanded into two  $\tau^\diamond$ -interpretations  $I_{V^\diamond}$  and  $I_{\neg V^\diamond}$  as follows:

$$I_{V^\diamond} = (I[V^\diamond/\mathbf{t}])^{D^\diamond} \text{ and } I_{\neg V^\diamond} = (I[V^\diamond/\mathbf{f}])^{D^\diamond}.$$

Since  $D^\diamond$  is a positive definition, hence total definition with open symbols  $\tau \cup V^\diamond$ , both interpretations are well-defined. Moreover they obviously satisfy:

$$I_{V^\diamond} \models \bigwedge V^\diamond \wedge D^\diamond \wedge \bigwedge \Gamma \wedge \neg \bigvee \Delta \text{ and } I_{\neg V^\diamond} \models \bigwedge \neg V^\diamond \wedge D^\diamond \wedge \bigwedge \Gamma \wedge \neg \bigvee \Delta.$$

By (2), it follows that

$$I_{V^\diamond} \models \bigwedge \neg V^\triangleright \text{ and } I_{\neg V^\diamond} \models \bigwedge V^\triangleright. \quad (3)$$

Let  $(I^n)_{n \leq \xi}$  be a terminal well-founded induction for  $D$  with limit  $I^\xi = I$ . There exists a least ordinal  $n$  such that  $P^{I^n} = \mathbf{u}$  for every  $P \in V$  and there exists at least one  $P \in V$  with  $P^{I^{n+1}} \neq \mathbf{u}$ . We distinguish between the case where  $P$  is  $\mathbf{t}$  in  $I^{n+1}$  and the case where  $P \in U$  for some non-empty set  $U \subseteq \tau_D^d$  such that all atoms of  $U$  are  $\mathbf{f}$  in  $I^{n+1}$ . We will prove in the first case that  $I_{V^\diamond} \models P^\triangleright$  and in the second case that  $I_{\neg V^\diamond} \models \neg P^\triangleright$  for every  $P^\triangleright \in V^\triangleright \cap U^\triangleright$ . This contradicts (3).

- Assume that  $P^{I^n} = \mathbf{u}$  and  $P^{I^{n+1}} = \mathbf{t}$ . Then for the rule  $P \leftarrow \varphi_P \in D$ , it holds that  $\varphi_P^{I^n} = \mathbf{t}$ . Consider the corresponding rule  $P^\triangleright \leftarrow \varphi_P^\diamond \in D^\diamond$ . If we can show that  $I_{V^\diamond} \models \varphi_P^\diamond$ , then Lemma 1 will yield that  $I_{V^\diamond} \models P^\triangleright$  which is what we must prove here.

Consider the  $\tau^\diamond$ -interpretation  $I^{n^\diamond}$  which extends  $I^n$  by interpreting each symbol  $Q^\triangleright$  and  $Q^\diamond$  as  $Q^{I^n}$ , i.e., as  $\mathbf{u}$ . Clearly,  $(\varphi_P^\diamond)^{I^{n^\diamond}} = \varphi_P^{I^n} = \mathbf{t}$ , and it suffices to show that  $I^{n^\diamond} \leq_p I_{V^\diamond}$  to obtain that  $I_{V^\diamond} \models \varphi_P^\diamond$ . But this is straightforward since  $I^{n^\diamond}|_\tau = I^n \leq_p I = I_{V^\diamond}|_\tau$  and  $(Q^\triangleright)^{I^{n^\diamond}} = (Q^\diamond)^{I^{n^\diamond}} = \mathbf{u}$  for each  $Q \in V$ . Hence, it is indeed the case that  $I^{n^\diamond} \leq_p I_{V^\diamond}$  which leads to the contradiction.

- For the other case, assume that  $I^{n+1} = I^n[U/\mathbf{f}]$  where  $P \in U$ . For each  $P \in U \cap V (\neq \emptyset)$  and its rule  $P \leftarrow \varphi_P \in D$ , it holds that  $\varphi_P^{I^{n+1}} = \mathbf{f}$ . We will use this to show that for each rule  $P^\triangleright \leftarrow \varphi_P^\diamond \in D^\diamond$  with  $P^\triangleright \in U^\triangleright \cap V^\triangleright$ ,  $\varphi_P^\diamond$  is false in the interpretation  $I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]$ . Then, since  $I_{\neg V^\diamond}$  satisfies  $D^\diamond$ , we can apply Lemma 4 to obtain that each  $P^\triangleright \in U^\triangleright \cap V^\triangleright$  is false in  $I_{\neg V^\diamond}$ . This produces the contradiction with (3).

The key point is therefore to show that all these renamed rule bodies  $\varphi_P^\diamond$  are false in the interpretation  $I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]$ . To prove this, we use the same technique as in the previous case, namely we construct an interpretation which is less precise than  $I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]$  and which falsifies all the concerned rule bodies. We choose this interpretation as the  $\tau^\diamond$ -interpretation  $I^\diamond$  which extends  $I^{n+1}$  by interpreting each symbol  $Q^\triangleright$  and  $Q^\diamond$  as  $Q^{I^{n+1}}$ , i.e. as  $\mathbf{f}$  if  $Q \in U \cap V$  and as  $\mathbf{u}$  if  $Q \in V \setminus U$ . Notice that for all formulas  $\psi$  over  $\tau$ , it holds that  $\psi^{I^{n+1}} = (\psi^\diamond)^{I^\diamond}$ .

Let us verify that  $I^\diamond \leq_p I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]$ . We have  $I^\diamond|_\tau = I^{n+1} \leq_p I = I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]|_\tau$ . The interpretation  $I^\diamond$  interprets all symbols  $Q^\diamond$  as  $\mathbf{u}$  or  $\mathbf{f}$  whereas  $I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]$  interprets them as  $\mathbf{f}$ , just like  $I_{\neg V^\diamond}$ . Symbols of  $U^\triangleright \cap V^\triangleright$  are interpreted as  $\mathbf{f}$  in both interpretations, and finally, the remaining symbols of  $V^\triangleright \setminus U^\triangleright$  are interpreted as  $\mathbf{u}$  in  $I^\diamond$  which is certainly less precise than in the other interpretation.

It follows that for every rule  $P^\triangleright \leftarrow \varphi_P^\diamond \in D^\diamond$  with  $P \in V \cap U$ ,  $\mathbf{f} = \varphi_P^{I^{n+1}} = (\varphi_P^\diamond)^{I^\diamond} \leq_p (\varphi_P^\diamond)^{I_{\neg V^\diamond}[U^\triangleright \cap V^\triangleright/\mathbf{f}]} = \mathbf{f}$ . As explained before, this leads to the desired contradiction.

**Lemma 6 (Soundness of the definition introduction rule).** *Let  $D$  be a total definition. If  $\models D', \Gamma \rightarrow \Delta, P' \equiv P$  for every  $P \in \tau_D^d$ , then  $\models \Gamma \rightarrow \Delta, D$ .*

*Proof.* Assume  $\models D', \Gamma \rightarrow \Delta, P' \equiv P$  for every  $P \in \tau_D^d$  but  $\not\models \Gamma \rightarrow \Delta, D$ . Then there exists a two-valued  $\tau$ -interpretation  $I$  such that  $I \models \bigwedge \Gamma$  but  $I \not\models \bigvee \Delta$ ,  $I \not\models D$ . Denote by  $J$  the two-valued well-founded model of  $D$  extending  $I|_{\tau_D^o}$ . Because  $I \not\models D$ , there exists a defined atom  $Q$  of  $D$  such that  $Q^I \neq Q^J$ . Since  $D$  is a total definition and  $D'$  is obtained by replacing each occurrence of each defined atom  $P$  in  $D$  by  $P'$ ,  $D'$  is a total definition. Thus, there exists a two-valued  $\tau'$ -interpretation  $I'$  such that  $I'$  is the well-founded model of  $D'$  extending  $I$ . Notice that for every  $P \in \tau_D^d$ ,  $P^{I'} = P^I$ . Because neither  $\Gamma$  nor  $\Delta$  contains an occurrence of an atom  $P'$ , it holds that  $I' \models \bigwedge \Gamma$  and  $I' \not\models \bigvee \Delta$ . Therefore, by the first assumption, it is obtained that  $I' \models P' \equiv P$  for every  $P \in \tau_D^d$ . Also, because  $D'$  is obtained by renaming all defined atoms and none of the open atoms, it holds that  $P^J = (P')^{I'}$  for every  $P \in \tau_D^d$ . Hence,  $Q^I = Q^{I'} = (Q')^{I'} = Q^J$ , a contradiction. Therefore,  $\models \Gamma \rightarrow \Delta, D$ .

The definition introduction rule is not sound if the inferred definition  $D$  is not total. We illustrate it with an example.

*Example 10.* Consider the definition as follows:

$$D = \{ P \leftarrow \neg P \}.$$

Let  $\Gamma$  and  $\Delta$  be empty sets. It is obvious that  $D' = \{ P' \leftarrow \neg P' \}$  is not total. Thus,  $\models D' \rightarrow P' \equiv P$  but  $\not\models \rightarrow D$ , which shows that the definition introduction rule is not sound when the inferred definition  $D$  is non-total.

Notice that all inference rules in **LPC(ID)** except the definition introduction rule are sound with respect to both total and non-total definitions. By induction on the number of inference rules in a proof of a sequent, we can easily prove the soundness of **LPC(ID)**.

**Theorem 1 (Soundness).** *If a sequent  $\Gamma \rightarrow \Delta$  is provable in **LPC(ID)** without using the definition introduction rule, then  $\models \Gamma \rightarrow \Delta$ . If a sequent  $\Gamma \rightarrow \Delta$  is provable in **LPC(ID)** and all definitions occurring in  $\Gamma$  and  $\Delta$  are total, then  $\models \Gamma \rightarrow \Delta$ .*

## 4.2 Completeness

**LPC(ID)** is not complete in general. Intuitively, this is because the only inference rules that allow to introduce a positive occurrence of a definition in the succedent of a sequent are the axioms, the weakening rules and the definition introduction rule. As shown in the above subsection, the definition introduction rule is not sound with respect to non-total definitions. Thus, no other inference rule allows to derive a non-total definition from some propositional formulas. Therefore, one cannot synthesize non-total definitions with **LPC(ID)**, i.e., not all valid sequents of the form  $\Gamma \rightarrow D$ , where  $D$  is a non-total definition, can be proved in this system.

We will however prove the completeness for a restricted class of sequents, namely the sequents  $\Gamma \rightarrow \Delta$  such that every definition occurring negatively in  $\Gamma$  or positively in  $\Delta$  must be total. The main difficulty in the completeness proof

for **LPC(ID)** is to handle the definitions in the sequents (We already know that the propositional part of **LPC(ID)** is complete. See e.g. [36,37]).

First, we focus on the completeness of sequents of the form  $D, \Gamma \rightarrow \Delta$ , where  $\Gamma$  and  $\Delta$  are sets of PC-formulas and  $D$  is a definition. Notice that the definition  $D$  appearing in the sequent may be non-total.

**Lemma 7.** *Let  $D$  be a definition and  $\Gamma$  a set of open literals of  $D$  such that for every  $Q \in \tau_D^0$  either  $Q \in \Gamma$  or  $\neg Q \in \Gamma$ . Let  $I_O$  be the unique two-valued  $\tau_D^0$ -interpretation such that  $I_O \models \bigwedge \Gamma$  and  $I$  the well-founded model of  $D$  extending  $I_O$ . If  $L$  is a defined literal of  $D$  such that  $L^I = \mathbf{t}$ , then  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.*

*Proof.* Let  $(I^n)_{n \leq \xi}$  be a terminal well-founded induction for  $D$  extending  $I_O$  with the limit  $I^\xi = I$ . Denote by  $\Delta^n$  a set of all defined literals  $L$  such that  $L^{I^n} = \mathbf{t}$  in arbitrary order. We prove that  $\Delta^n, D, \Gamma \rightarrow L$  is provable in **LPC(ID)** for all  $L \in \Delta^{n+1} \setminus \Delta^n$ . For each  $L \in \Delta^{n+1} \setminus \Delta^n$ ,  $L^{I^n} = \mathbf{u}$  and  $L^{I^{n+1}} = \mathbf{t}$ . We distinguish between the case where  $\Delta^{n+1} \setminus \Delta^n$  contains one positive literal and the case where it contains a set of negative literals.

- Assume that  $\Delta^{n+1} \setminus \Delta^n$  consists of one defined atom  $P$ . For every two-valued  $\tau$ -interpretation  $J$  such that  $J$  is a model of  $\bigwedge \Gamma$  and  $\bigwedge \Delta^n$ ,  $I^n \leq_p J$ . Indeed,  $L^{I^n} = L^J = \mathbf{t}$  for every  $L \in \Gamma$ ,  $L^{I^n} = L^J = \mathbf{t}$  for every  $L \in \Delta^n$  and for every other atom  $Q \in \tau$ ,  $Q^{I^n} = \mathbf{u} \leq_p Q^J$ .  $P^{I^{n+1}} = \mathbf{t}$ , hence  $\varphi_P^{I^n} = \mathbf{t}$ . It follows that  $\varphi_P^J = \mathbf{t}$ . Thus,  $\models \Delta^n, \Gamma \rightarrow \varphi_P$ . Therefore, by the completeness of the propositional part of **LPC(ID)**, the sequent  $\Delta^n, \Gamma \rightarrow \varphi_P$  is provable in **LPC(ID)**. Hence, by the right definition rule,  $\Delta^n, D, \Gamma \rightarrow P$  is provable in **LPC(ID)**.
- For the other case, assume that  $\Delta^{n+1} \setminus \Delta^n$  is a set of negative literals. Denote the set  $\{P \mid \neg P \in \Delta^{n+1} \setminus \Delta^n\}$  by  $U$ . Recall that  $I^{n+1} = I^n[U/\mathbf{f}]$ .  $P^{I^{n+1}} = \mathbf{f}$  for each  $P \in U$ , hence  $\varphi_P^{I^{n+1}} = \mathbf{f}$ . Consider the interpretation  $I^{n+1^\triangleright} = I^n[U^\triangleright/\mathbf{f}]$ . There are two simple observations that can be made about  $I^{n+1^\triangleright}$  and each  $\varphi_P^\triangleright$ :
  - $I^{n+1^\triangleright} \leq_p J'$  for every two-valued  $\tau \cup U^\triangleright$ -interpretation  $J'$  such that  $J'$  satisfies  $\bigwedge \Gamma$ ,  $\bigwedge \Delta^n$  and  $\bigwedge \neg U^\triangleright$ : indeed,  $L^{I^{n+1^\triangleright}} = L^{J'} = \mathbf{t}$  for every  $L \in \Gamma$ ,  $L^{I^{n+1^\triangleright}} = L^{J'} = \mathbf{t}$  for every  $L \in \Delta^n$ ,  $(P^\triangleright)^{I^{n+1^\triangleright}} = P^\triangleright^{J'} = \mathbf{f}$  for every  $P^\triangleright \in U^\triangleright$  and  $Q^{I^{n+1^\triangleright}} = \mathbf{u} \leq_p Q^{J'}$  for every other atom  $Q \in \tau \cup U^\triangleright$ .
  - $(\varphi_P^\triangleright)^{I^{n+1^\triangleright}} = \varphi_P^{I^{n+1}} = \mathbf{f}$ : obvious from the construction of  $I^{n+1^\triangleright}$  and  $\varphi_P^\triangleright$ .
 Combining these results, we obtain  $(\varphi_P^\triangleright)^{J'} = \mathbf{f}$  for every two-valued interpretation  $J'$  satisfying  $\bigwedge \Gamma$ ,  $\bigwedge \Delta^n$  and  $\bigwedge \neg U^\triangleright$ . It follows that  $\models \neg U^\triangleright, \Delta^n, \Gamma \rightarrow \neg \varphi_P^\triangleright$  for every  $P \in U$ . By the completeness of the propositional part of **LPC(ID)**, the left definition rule and the right  $\neg$  rule the sequent  $\Delta^n, D, \Gamma \rightarrow \neg P$  is provable in **LPC(ID)** for every  $P \in U$ .

Since  $(I^n)_{n \leq \xi}$  is a terminal well-founded induction for  $D$  with the limit  $I = I^\xi$ , it is obvious that the set of defined literals  $L$  for which  $L^I = \mathbf{t}$  is exactly the set of all defined literals in  $\Delta^\xi$ . Thus, by using the cut rule, it is easy to show by induction on  $n$  that if  $L$  is a defined literal of  $D$  such that  $L^I = \mathbf{t}$ , the sequent  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.

Notice that in the above lemma, we do not require the totality of the definition. So the definition  $D$  can be non-total and the well-founded model of  $D$  may be a three-valued interpretation.

**Lemma 8.** *Let  $D$  be a total definition and let  $\Gamma$  be a set of open literals of  $D$ , such that for every atom  $Q \in \tau_D^o$  either  $Q \in \Gamma$  or  $\neg Q \in \Gamma$ . Let  $L$  be a defined literal of  $D$ . If  $\models D, \Gamma \rightarrow L$ , then  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.*

*Proof.* Assume that  $\models D, \Gamma \rightarrow L$ . Let  $I_O$  be the unique two-valued  $\tau_D^o$ -interpretation such that  $I_O \models \bigwedge \Gamma$ . Because  $D$  is total,  $I_O$  can be extended to a two-valued well-founded model  $I$  of  $D$  such that  $I \models \bigwedge \Gamma$  and  $I \models D$ . Then since  $\models D, \Gamma \rightarrow L$ , it holds that  $L^I = \mathbf{t}$ . Thus, by Lemma 7,  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.

**Lemma 9.** *Let  $D$  be a total definition and  $\Gamma$  an arbitrary consistent set of literals. If  $L$  is a defined literal of  $D$  such that  $\models D, \Gamma \rightarrow L$ , then  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.*

*Proof.* Let  $\Gamma'$  be an arbitrary extension of  $\Gamma$  such that for every open atom  $Q$  of  $D$ , either  $Q \in \Gamma'$  or  $\neg Q \in \Gamma'$ . First, we want to show that  $D, \Gamma' \rightarrow L$  is provable in **LPC(ID)**. It holds that  $\models D, \Gamma' \rightarrow L$  because  $\models D, \Gamma \rightarrow L$ . Consider the set  $\Gamma''$  of all open literals of  $D$  in  $\Gamma'$ . If  $\models D, \Gamma'' \rightarrow L$ , then by the previous lemma,  $D, \Gamma'' \rightarrow L$  is provable in **LPC(ID)**, and by the left weakening rule, so is  $D, \Gamma' \rightarrow L$ . If  $\not\models D, \Gamma'' \rightarrow L$ , then by totality of  $D$ ,  $\models D, \Gamma'' \rightarrow \neg L$  and hence,  $\models D, \Gamma' \rightarrow \neg L$ . This means that  $D \wedge \bigwedge \Gamma'$  is unsatisfiable, which implies that for some defined literal  $L'$  in  $\Gamma'$ ,  $\models D, \Gamma'' \rightarrow \neg L'$ . By the previous lemma and the left weakening rule,  $D, \Gamma' \rightarrow \neg L'$  is provable in **LPC(ID)**. It is obvious that  $D, \Gamma' \rightarrow L'$  is an axiom because  $L'$  is a literal in  $\Gamma'$ . Then we can use the left  $\neg$  rule, the cut rule and the right weakening rule to show that  $D, \Gamma' \rightarrow L$  is provable in **LPC(ID)**.

Given that the sequents  $D, \Gamma' \rightarrow L$  are provable in **LPC(ID)** for all extensions  $\Gamma'$  of  $\Gamma$ , by using the right  $\neg$  rule and the cut rule on all  $D, \Gamma' \rightarrow L$ , an **LPC(ID)**-proof for  $D, \Gamma \rightarrow L$  can be constructed.

**Lemma 10.** *Let  $D$  be a definition and  $\Gamma$  a set of open literals of  $D$ , such that for every atom  $Q \in \tau_D^o$  either  $Q \in \Gamma$  or  $\neg Q \in \Gamma$ . If  $\models D, \Gamma \rightarrow \perp$ , then  $D, \Gamma \rightarrow \perp$  is provable in **LPC(ID)**.*

*Proof.* Let  $I_O$  be the unique two-valued  $\tau_D^o$ -interpretation such that  $I_O \models \bigwedge \Gamma$  and  $(I^n)_{n \leq \xi}$  a terminal well-founded induction for  $D$  extending  $I_O$  with limit  $I^\xi = I$ . Because  $\models D, \Gamma \rightarrow \perp$ , there is no two-valued well-founded model for  $D$  extending  $I_O$ . Hence  $I$  is a three-valued  $\tau$ -interpretation. Denote by  $E$  the set of all defined atoms of  $D$  which are not unknown in  $I$  and  $V$  the set  $\tau_D^d \setminus E$ . For each  $P \in E$ , we define a literal  $L_P$  as follows:

$$L_P = \begin{cases} P & \text{if } P^I = \mathbf{t} \\ \neg P & \text{if } P^I = \mathbf{f} \end{cases}.$$

Denote by  $K$  the set  $\{L_P \mid P \in E\}$  of literals. We first want to show that

$$\models D^\diamond, V^\diamond, K, \Gamma \rightarrow \bigwedge \neg V^\triangleright \text{ and } \models D^\diamond, \neg V^\diamond, K, \Gamma \rightarrow \bigwedge V^\triangleright. \quad (4)$$

Consider the vocabulary  $\tau^\diamond = \tau \cup V^\triangleright \cup V^\diamond$ .  $I$  can be expanded into two  $\tau^\diamond$ -interpretations  $I_{V^\diamond}$  and  $I_{\neg V^\diamond}$  as follows:

$$I_{V^\diamond} = (I[V^\diamond/\mathbf{t}])^{D^\diamond} \quad \text{and} \quad I_{\neg V^\diamond} = (I[V^\diamond/\mathbf{f}])^{D^\diamond}.$$

Since  $D^\diamond$  is a positive definition, hence total definition with open symbols  $\tau \cup V^\diamond$ , both interpretations are well-defined. Moreover it is obvious that  $I_{V^\diamond}$ , respectively  $I_{\neg V^\diamond}$ , is the only interpretation satisfying:

$$I_{V^\diamond} \models D^\diamond \wedge \bigwedge V^\diamond \wedge \bigwedge K \wedge \bigwedge \Gamma, \quad \text{respectively} \quad I_{\neg V^\diamond} \models D^\diamond \wedge \bigwedge \neg V^\diamond \wedge \bigwedge K \wedge \bigwedge \Gamma.$$

In order to prove (4), it suffices to show that

$$I_{V^\diamond} \models \bigwedge \neg V^\triangleright \quad \text{and} \quad I_{\neg V^\diamond} \models \bigwedge V^\triangleright. \quad (5)$$

- We want to prove that  $I_{V^\diamond} \models \bigwedge \neg V^\triangleright$ . For any  $P \in V$  with its rule  $P \leftarrow \varphi_P \in D$ ,  $P^\triangleright \leftarrow \varphi_P^\diamond$  is the corresponding rule for  $P^\triangleright$  in  $D^\diamond$ . If we can show that  $I_{V^\diamond}[V^\triangleright/\mathbf{f}] \models \neg \varphi_P^\diamond$  for each  $P^\triangleright \in V^\triangleright$  with its rule  $P^\triangleright \leftarrow \varphi_P^\diamond$ , then since  $I_{V^\diamond}$  satisfies  $D^\diamond$ , we can apply Lemma 4 to obtain that each  $P^\triangleright \in V^\triangleright$  is false in  $I_{V^\diamond}$ , which is what we must prove here.

Consider the  $\tau^\diamond$ -interpretation  $I^\diamond$  which extends  $I$  by interpreting each symbol  $Q^\triangleright$  and  $Q^\diamond$  as  $Q^I$  for each  $Q \in V$ , i.e., as  $\mathbf{u}$ . Clearly, for every  $P \in V$  with its rule  $P \leftarrow \varphi_P \in D$ ,  $(\varphi_P^\diamond)^{I^\diamond} = \varphi_P^I = \mathbf{u}$ , and since  $(\varphi_P^\diamond)^{I_{V^\diamond}[V^\triangleright/\mathbf{f}]} \neq \mathbf{u}$ , it is sufficient to show that  $(\varphi_P^\diamond)^{I_{V^\diamond}[V^\triangleright/\mathbf{f}]} \leq (\varphi_P^\diamond)^{I^\diamond}$  to obtain that  $I_{V^\diamond}[V^\triangleright/\mathbf{f}] \models \neg \varphi_P^\diamond$  for every  $P^\triangleright \in V^\triangleright$  with its rule  $P^\triangleright \leftarrow \varphi_P^\diamond \in D^\diamond$ . This can be verified by the following observations.

- $I_{V^\diamond}[V^\triangleright/\mathbf{f}]|_\tau = I^\diamond|_\tau$ .
- For every  $Q \in V$ , every occurrence of  $Q^\triangleright$  in  $\varphi_P^\diamond$  is positive and  $(Q^\triangleright)^{I_{V^\diamond}[V^\triangleright/\mathbf{f}]} = \mathbf{f} \leq (Q^\triangleright)^{I^\diamond} = \mathbf{u}$ .
- For every  $Q \in V$ , every occurrence of  $Q^\diamond$  in  $\varphi_P^\diamond$  is negative and  $(Q^\diamond)^{I_{V^\diamond}[V^\triangleright/\mathbf{f}]} = \mathbf{t} \geq (Q^\diamond)^{I^\diamond} = \mathbf{u}$ .

Hence, it is indeed the case that  $(\varphi_P^\diamond)^{I_{V^\diamond}[V^\triangleright/\mathbf{f}]} \leq (\varphi_P^\diamond)^{I^\diamond}$ , as desired.

- We want to prove that  $I_{\neg V^\diamond} \models \bigwedge V^\triangleright$ . Assume toward contradiction that there exists a non-empty set  $F^\triangleright \subseteq V^\triangleright$  such that  $I_{\neg V^\diamond} \models \bigwedge \neg F^\triangleright$  and for the set  $T^\triangleright = V^\triangleright \setminus F^\triangleright$ ,  $I_{\neg V^\diamond} \models \bigwedge T^\triangleright$ . Consider the  $\tau$ -interpretation  $I^1 = I[F/\mathbf{f}]$ . If we can show that  $\varphi_P^{I^1} = \mathbf{f}$  for every  $P \in F$  with its rule  $P \leftarrow \varphi_P \in D$ , then since for each  $P \in F$  and its rule  $P \leftarrow \varphi_P \in D$ ,  $P^I = \mathbf{u}$  and  $\varphi_P^{I^1} = \mathbf{f}$ ,  $I$  can be extended to  $I^1$  in the well-founded induction  $(I^n)_{n \leq \xi}$  for  $D$ . This produces the contradiction to that  $I$  is the limit of  $(I^n)_{n \leq \xi}$ . To prove that  $\varphi_P^{I^1} = \mathbf{f}$  for every  $P \in F$  with the rule  $P \leftarrow \varphi_P \in D$ , we first choose a  $\tau^\diamond$ -interpretation  $I^\diamond$  which extends  $I^1$  by interpreting each symbol  $Q^\triangleright$  and  $Q^\diamond$  as  $Q^{I^1}$ , i.e., as  $\mathbf{f}$  if  $Q \in F$  and as  $\mathbf{u}$  if  $Q \in T$ . Notice that for all formulas  $\psi$  over  $\tau$ , it holds that  $\psi^{I^1} = (\psi^\diamond)^{I^\diamond}$ . Thus, it is sufficient to show that  $(\varphi_P^\diamond)^{I^\diamond} = \mathbf{f}$  for every  $P^\triangleright \in F^\triangleright$  with the rule  $P^\triangleright \leftarrow \varphi_P^\diamond \in D^\diamond$ . Since  $I_{\neg V^\diamond} \models \neg P^\triangleright$  for each  $P^\triangleright \in F^\triangleright$  and  $I_{\neg V^\diamond}$  is a model of  $D^\diamond$ , by Lemma 1, we have that  $(\varphi_P^\diamond)^{I_{\neg V^\diamond}} = \mathbf{f}$  for every  $P^\triangleright \in F^\triangleright$  with the rule  $P^\triangleright \leftarrow \varphi_P^\diamond \in D^\diamond$ . If we can have that  $(\varphi_P^\diamond)^{I^\diamond} \leq (\varphi_P^\diamond)^{I_{\neg V^\diamond}} = \mathbf{f}$ , it holds that  $(\varphi_P^\diamond)^{I^\diamond} = \mathbf{f}$ , which is exactly what we need.

We can verify that  $(\varphi_P^\diamond)^{I^\diamond} \leq (\varphi_P^\diamond)^{I_{\neg V^\diamond}}$  by the following facts.



- $I_{\neg V^\diamond}|_\tau = I^\diamond|_\tau$ .
  - Every occurrence of  $Q^\triangleright$  in  $\varphi_P^\diamond$  is positive and  $(Q^\triangleright)^{I_{\neg V^\diamond}} = (Q^\triangleright)^{I^\diamond} = \mathbf{f}$  for each  $Q^\triangleright \in F^\triangleright$  while  $(Q^\triangleright)^{I^\diamond} = \mathbf{u} \leq (Q^\triangleright)^{I_{\neg V^\diamond}} = \mathbf{t}$  for each  $Q^\triangleright \in V^\triangleright \setminus F^\triangleright$ .
  - Every occurrence of  $Q^\diamond$  in  $\varphi_P^\diamond$  is negative and  $(Q^\diamond)^{I_{\neg V^\diamond}} = (Q^\diamond)^{I^\diamond} = \mathbf{f}$  for each  $Q^\diamond \in F^\diamond$  while  $(Q^\diamond)^{I^\diamond} = \mathbf{u} \geq (Q^\diamond)^{I_{\neg V^\diamond}} = \mathbf{f}$  for each  $Q^\diamond \in V^\diamond \setminus F^\diamond$ .
- Hence, it is the case that  $(\varphi_P^\diamond)^{I^\diamond} \leq (\varphi_P^\diamond)^{I_{\neg V^\diamond}} = \mathbf{f}$ , as desired.

Therefore, it is obtained that  $\models D^\diamond, V^\diamond, K, \Gamma \rightarrow \bigwedge \neg V^\triangleright$  and  $\models D^\diamond, \neg V^\diamond, K, \Gamma \rightarrow \bigwedge V^\triangleright$ .  $D^\diamond$  is a total definition, hence by using Lemma 9 and the right  $\wedge$  rule, both  $V^\diamond, D^\diamond, K, \Gamma \rightarrow \bigwedge \neg V^\triangleright$  and  $\neg V^\diamond, D^\diamond, K, \Gamma \rightarrow \bigwedge V^\triangleright$  are provable in **LPC(ID)**. It follows from the non-total definition rule that  $K, D, \Gamma \rightarrow \perp$  is provable in **LPC(ID)**. Since  $I$  is a well-founded model of  $D$  extending  $I_O$  and  $L^I = \mathbf{t}$  for each  $L \in K$ , using Lemma 7, it holds that for each  $L \in K$ ,  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**. Consequently, by the multiple use of the cut rule on  $K, D, \Gamma \rightarrow \perp$  and  $D, \Gamma \rightarrow L$  for each  $L \in K$ ,  $D, \Gamma \rightarrow \perp$  is provable in **LPC(ID)**.

**Lemma 11.** *Let  $D$  be a definition and  $\Gamma$  a set of open literals of  $D$  such that for every atom  $Q \in \tau_D^\triangleright$ , either  $Q \in \Gamma$  or  $\neg Q \in \Gamma$ . Let  $L$  be a defined literal of  $D$ . If  $\models D, \Gamma \rightarrow L$ , then  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.*

*Proof.* Assume  $\models D, \Gamma \rightarrow L$ . Let  $I_O$  be the unique two-valued  $\tau_D^\triangleright$ -interpretation such that  $I_O \models \bigwedge \Gamma$ . If  $\not\models D, \Gamma \rightarrow \perp$ , then  $I_O$  can be extended to the two-valued well-founded model  $I$  of  $D$  such that  $I \models \bigwedge \Gamma$  and  $I \models D$ . Since  $\models D, \Gamma \rightarrow L$ , it holds that  $I \models L$ . Thus, by Lemma 7,  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**. If  $\models D, \Gamma \rightarrow \perp$ , then by Lemma 10,  $D, \Gamma \rightarrow \perp$  is provable in **LPC(ID)**. Hence, by the right weakening rule,  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.

**Lemma 12.** *Let  $D$  be a definition and  $\Gamma$  an arbitrary consistent set of literals. If  $\models D, \Gamma \rightarrow \perp$ , then  $D, \Gamma \rightarrow \perp$  is provable in **LPC(ID)**.*

To prove this, we use the same technique as in the proof of Lemma 9. We omit the details of the proof here.

**Lemma 13.** *Let  $D$  be a definition,  $\Gamma$  an arbitrary consistent set of literals and  $L$  a defined literal of  $D$ . If  $\models D, \Gamma \rightarrow L$ , then  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.*

*Proof.* If  $\Gamma \cup \{\neg L\}$  is an inconsistent set of literals, we have that  $D, \Gamma \rightarrow L$  is an axiom and thus,  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**. If  $\Gamma \cup \{\neg L\}$  is consistent, because  $\models D, \Gamma, \neg L \rightarrow \perp$ , by the previous lemma, it is obtained that  $D, \Gamma, \neg L \rightarrow \perp$  is provable in **LPC(ID)**. Then by the  $\neg$  rules and the cut rule, we can conclude that  $D, \Gamma \rightarrow L$  is provable in **LPC(ID)**.

The remainder of the completeness proof for the class of sequents, namely the sequents  $\Gamma, D \rightarrow \Delta$  where  $\Gamma$  and  $\Delta$  are sets of PC-formulas and  $D$  is a definition, will use a standard technique: we construct the so called *reduction tree* for a sequent  $\Gamma \rightarrow \Delta$ . We follow the approach from [37]. First, we introduce some terminology.

**Definition 5.** *A reduction tree for a sequent  $S = \Gamma \rightarrow \Delta$  is a tree  $T_S$  of sequents. The root of  $T_S$  is  $S$ . Moreover,  $T_S$  is constructed by applying one of the following reductions on each non-leaf  $\Pi \rightarrow \Sigma$ .*

- (left  $\neg$  reduction)  $\Pi$  contains a sequent formula  $\neg A$ , then write down  $\Pi \setminus \{\neg A\} \rightarrow \Sigma, A$  as the unique child of  $\Pi \rightarrow \Sigma$ .
- (right  $\neg$  reduction)  $\Sigma$  contains a sequent formula  $\neg A$ , then write down  $A, \Pi \rightarrow \Sigma \setminus \{\neg A\}$  as the unique child of  $\Pi \rightarrow \Sigma$ .
- (left  $\wedge$  reduction)  $\Pi$  contains a sequent formula  $A \wedge B$ , then write down  $A, B, \Pi \setminus \{A \wedge B\} \rightarrow \Sigma$  as the unique child of  $\Pi \rightarrow \Sigma$ .
- (right  $\wedge$  reduction)  $\Sigma$  contains a sequent formula  $A \wedge B$ , then write down  $\Pi \rightarrow \Sigma \setminus \{A \wedge B\}, A$  and  $\Pi \rightarrow \Sigma \setminus \{A \wedge B\}, B$  as two children of  $\Pi \rightarrow \Sigma$ .
- (left  $\vee$  reduction)  $\Pi$  contains a sequent formula  $A \vee B$ , then write down  $A, \Pi \setminus \{A \vee B\} \rightarrow \Sigma$  and  $B, \Pi \setminus \{A \vee B\} \rightarrow \Sigma$  as two children of  $\Pi \rightarrow \Sigma$ .
- (right  $\vee$  reduction)  $\Sigma$  contains a sequent formula  $A \vee B$ , then write down  $\Pi \rightarrow \Sigma \setminus \{A \vee B\}, A, B$  as the unique child of  $\Pi \rightarrow \Sigma$ .
- (definition introduction reduction)  $\Sigma$  contains a sequent formula  $D$ , which is a total definition with  $\tau_D^d = \{P_1, \dots, P_n\}$ , then write down  $D', \Pi \rightarrow \Sigma \setminus \{D\}, P'_i \equiv P_i$  for each  $i \in [1, n]$  as  $n$  children of  $\Pi \rightarrow \Sigma$ .

In addition, each leaf of  $T_S$  is either an axiom, or none of the above reductions is possible.

Observe that the definition introduction reduction corresponds to the definition introduction rule while each other reduction respectively corresponds to a logical inference rule. Each leaf node of a reduction tree is either an axiom or a sequent of the form  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$  where  $\Gamma$  and  $\Delta$  are sets of atoms with  $\Gamma \cap \Delta = \emptyset$  and  $D_1, \dots, D_n$  are definitions.

**Definition 6.** An inference rule preserves counter-model if for each instance of the inference rule, a counter-model for one of the premises of the instance is the same as a counter-model for the conclusion of the instance.

The following property can easily be verified.

**Proposition 4.** All the logical inference rules preserve counter-models.

**Lemma 14.** The definition introduction rule preserves counter-model.

*Proof.* Let  $D$  be a total definition. Then  $D'$  is a total definition because of its construction. Assume that  $I$  is a counter-model of  $D', \Gamma \rightarrow \Delta, P' \equiv P$  for some  $P \in \tau_D^d$ , but  $I$  is not a counter-model of  $\Gamma \rightarrow \Delta, D$ . Since  $D$  and  $D'$  are total,  $I$  is a two-valued interpretation satisfying  $D', \bigwedge \Gamma, \neg \bigvee \Delta$  and  $\neg(P' \equiv P)$ . Because  $I$  is not a counter-model for  $\Gamma \rightarrow \Delta, D$ , it holds that  $I \models D$ . Obviously from the construction of  $D'$  and the fact that  $I$  satisfies both  $D$  and  $D'$ , we conclude that  $I \models P' \equiv P$  for every  $P \in \tau_D^d$ , a contradiction.

Then we obtain the property of reduction trees as follows.

**Proposition 5.** For each sequent  $S = \Gamma \rightarrow \Delta$ , (a) there exists a reduction tree  $T_S$ , (b) if all leaf nodes of a reduction tree  $T_S$  are provable in **LPC(ID)**, then the root sequent is provable in **LPC(ID)**, and (c), there exists a leaf node of  $T_S$  such that a counter-model for this leaf node is a counter-model for the root.

*Proof.* Clearly, a reduction tree exists because it can be constructed by a non-deterministic reduction process. Because each reduction in a reduction tree corresponds to either the definition introduction rule or a logical inference rule, by using the corresponding inference rule, it is easy to prove that if the children of a node in a reduction tree are provable in **LPC(ID)**, then the node itself is provable in **LPC(ID)**. Therefore, the root sequent is provable in **LPC(ID)** if all leaf nodes of the reduction tree are provable in **LPC(ID)**.

A counter-model for a leaf is a counter-model for the root because all the logical inference rules and the definition introduction rule preserve counter-models by Proposition 4 and Lemma 14 and each non-leaf node can be proved from its children using only those inference rules.

We are now ready to prove the completeness theorem of the sequents of the form  $D, \Gamma \rightarrow \Delta$ , where  $\Gamma$  and  $\Delta$  are sets of PC-formulas and  $D$  is a definition.

**Theorem 2 (Completeness for one definition in the antecedent).** *Let  $\Gamma$  and  $\Delta$  be sets of PC-formulas and  $D$  a definition. If  $\models D, \Gamma \rightarrow \Delta$ , then  $D, \Gamma \rightarrow \Delta$  is provable in **LPC(ID)**.*

*Proof.* First, a reduction tree is constructed from the root  $D, \Gamma \rightarrow \Delta$ . Every leaf of the reduction tree must be an axiom or a sequent of the form  $D, \Pi \rightarrow \Sigma$ , where  $\Pi$  and  $\Sigma$  are (possibly empty) sets of propositional atoms satisfying that (a)  $\Pi$  and  $\Sigma$  have no atom in common, and (b) when  $\Sigma$  is not empty,  $\Pi$  or  $\Sigma$  contains at least one defined atom of  $D$ . By (c) of Proposition 5, if  $\models D, \Gamma \rightarrow \Delta$ , then  $\models D, \Pi \rightarrow \Sigma$ . Hence, if  $\Sigma$  is empty, by Lemma 12, it is obtained that  $D, \Pi \rightarrow \Sigma$  is provable in **LPC(ID)**. If  $\Sigma$  is not empty, by Lemma 13, the  $\neg$  rules and the weakening rules,  $D, \Pi \rightarrow \Sigma$  is provable in **LPC(ID)**. Extending for every leaf  $D, \Pi \rightarrow \Sigma$  the branch that ends in that leaf with the proof tree for that leaf, yields an **LPC(ID)**-proof for  $D, \Gamma \rightarrow \Delta$ .

**LPC(ID)** remains complete for sequents of the form  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$ , where  $\Gamma$  and  $\Delta$  are sets of PC-formulas and multiple definitions are allowed in the antecedent.

**Lemma 15.** *Let  $D_1, \dots, D_n$  be definitions and  $\Gamma$  an arbitrary consistent set of literals. If  $\models D_1, \dots, D_n, \Gamma \rightarrow \perp$ , then  $D_1, \dots, D_n, \Gamma \rightarrow \perp$  is provable in **LPC(ID)**.*

*Proof.* Let  $\Gamma'$  be an arbitrary extension of  $\Gamma$  such that for every  $D_i \in \{D_1, \dots, D_n\}$  and every open atom  $Q$  of  $D_i$ , either  $Q \in \Gamma'$  or  $\neg Q \in \Gamma'$ . First, we want to show that  $D_1, \dots, D_n, \Gamma' \rightarrow \perp$  is provable in **LPC(ID)**. It holds that  $\models D_1, \dots, D_n, \Gamma' \rightarrow \perp$  because  $\models D_1, \dots, D_n, \Gamma \rightarrow \perp$ . Consider the set  $\Gamma''$  of all open literals of all definitions  $D_1, \dots, D_n$  in  $\Gamma'$ . We distinguish between the case where  $\models D_1, \dots, D_n, \Gamma'' \rightarrow \perp$  and the case where  $\not\models D_1, \dots, D_n, \Gamma'' \rightarrow \perp$ .

- In the first case where  $\models D_1, \dots, D_n, \Gamma'' \rightarrow \perp$ , we distinguish between the subcase where there exists at least one  $D_i \in \{D_1, \dots, D_n\}$  such that  $\models D_i, \Gamma'' \rightarrow \perp$  and the subcase where for every  $D_i \in \{D_1, \dots, D_n\}$  it holds that  $\not\models D_i, \Gamma'' \rightarrow \perp$ .

- In the first subcase,  $\models D_i, \Gamma'' \rightarrow \perp$ , hence by Lemma 12,  $D_i, \Gamma'' \rightarrow \perp$  is provable in **LPC(ID)**. Then by using the left weakening rule, we conclude that  $D_1, \dots, D_n, \Gamma' \rightarrow \perp$  is provable in **LPC(ID)**.
  - In the other subcase, it holds that  $\not\models D_i, \Gamma'' \rightarrow \perp$  for every  $D_i \in \{D_1, \dots, D_n\}$ . Thus, for every  $D_i \in \{D_1, \dots, D_n\}$ , there exists a unique two-valued well-founded model  $I_i$  of  $D_i$  such that  $I_i \models D_i$  and  $I_i \models \bigwedge \Gamma''$ . Because  $D_1 \wedge \dots \wedge D_n \wedge \bigwedge \Gamma''$  is unsatisfiable, for some  $I_i$  and  $I_j$  such that  $i \neq j$  and for some defined literal  $L$ , it can be implied that  $I_i \models L$  and  $I_j \models \neg L$ . Thus, we have that  $\models D_i, \Gamma'' \rightarrow L$  and  $\models D_j, \Gamma'' \rightarrow \neg L$ . Therefore, by Lemma 13, it is concluded that both  $D_i, \Gamma'' \rightarrow L$  and  $D_j, \Gamma'' \rightarrow \neg L$  are provable in **LPC(ID)**. Then we can use the left weakening rule, the left  $\neg$  rule and the cut rule to show that  $D_1, \dots, D_n, \Gamma' \rightarrow \perp$  is provable in **LPC(ID)**.
- In the other case where  $\not\models D_1, \dots, D_n, \Gamma'' \rightarrow \perp$ , hence there exists a unique two-valued interpretation  $I$  such that  $I \models D_1 \wedge \dots \wedge D_n \wedge \bigwedge \Gamma''$ . Because  $\not\models D_1, \dots, D_n, \Gamma'' \rightarrow \perp$ , for each  $D_i \in \{D_1, \dots, D_n\}$ , it holds that  $\not\models D_i, \Gamma'' \rightarrow \perp$  and hence, there exists a unique two-valued well-founded model  $I_i$  of  $D_i$  such that  $I_i \models D_i$  and  $I_i \models \bigwedge \Gamma''$ . Therefore, for each  $D_i$  and each defined atom  $P \in \tau_{D_i}^d$ ,  $P^{I_i} = P^I$ . Since  $D_1 \wedge \dots \wedge D_n \wedge \bigwedge \Gamma''$  is satisfiable but  $D_1 \wedge \dots \wedge D_n \wedge \bigwedge \Gamma'$  is unsatisfiable, it can be implied that for some defined literal  $L'$  in  $\Gamma'$ ,  $\models D_1, \dots, D_n, \Gamma'' \rightarrow \neg L'$ . Assume that  $L'$  is a defined literal of  $D_i$ . Because  $L'^{I_i} = L'^I = \mathbf{f}$ , we have that  $\models D_i, \Gamma'' \rightarrow \neg L'$ . By Lemma 13 and the left weakening rule,  $D_i, \Gamma'' \rightarrow \neg L'$  is provable in **LPC(ID)**. It is obvious that  $D_i, \Gamma' \rightarrow L'$  is an axiom because  $L'$  is a literal in  $\Gamma'$ . Then we can use the left weakening rule, the left  $\neg$  rule and the cut rule to show that  $D_1, \dots, D_n, \Gamma' \rightarrow \perp$  is provable in **LPC(ID)**.

Given that the sequents  $D_1, \dots, D_n, \Gamma' \rightarrow \perp$  are provable in **LPC(ID)** for all extensions  $\Gamma'$  of  $\Gamma$ , by using the right  $\neg$  rule and the cut rule on all  $D_1, \dots, D_n, \Gamma' \rightarrow \perp$ , we can construct an **LPC(ID)**-proof for  $D_1, \dots, D_n, \Gamma \rightarrow \perp$ .

**Lemma 16.** *Let  $D_1, \dots, D_n$  be definitions and let  $\Gamma$  and  $\Delta$  be sets of atoms. If  $\models D_1, \dots, D_n, \Gamma \rightarrow \Delta$ , then  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$  is provable in **LPC(ID)**.*

*Proof.* The proof is trivial if  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$  is an axiom, hence we assume that  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$  is not an axiom, i.e.  $\Gamma \cap \Delta = \emptyset$ . Because  $\Gamma, \neg\Delta$  is a consistent set of literals and  $\models D_1, \dots, D_n, \Gamma, \neg\Delta \rightarrow \perp$ , by the previous lemma, we have that  $D_1, \dots, D_n, \Gamma, \neg\Delta \rightarrow \perp$  is provable in **LPC(ID)**. Then by the  $\neg$  rules and the cut rule, we can conclude that  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$  is provable in **LPC(ID)**.

The following completeness theorem of the sequents with multiple definitions in the antecedent is an immediate consequence of Lemma 16 and the reduction tree for sequents.

**Theorem 3 (Completeness for multiple definitions in the antecedent).**

*Let  $\Gamma$  and  $\Delta$  be sets of PC-formulas and  $D_1, \dots, D_n$  definitions. If  $\models D_1, \dots, D_n, \Gamma \rightarrow \Delta$ , then  $D_1, \dots, D_n, \Gamma \rightarrow \Delta$  is provable in **LPC(ID)**.*

Then we have the following main completeness theorem.

**Theorem 4 (Completeness).** *If  $\models \Gamma \rightarrow \Delta$  and all definitions occurring either negatively in  $\Gamma$  or positively in  $\Delta$  are total, then  $\Gamma \rightarrow \Delta$  is provable in **LPC(ID)**.*

*Proof.* Let  $\Gamma \rightarrow \Delta$  be a valid sequent such that any definition which occurs either negatively in  $\Gamma$  or positively in  $\Delta$  is total and let  $T_S$  be a reduction tree with root  $\Gamma \rightarrow \Delta$ . Then by (c) of Proposition 5, all leaves of  $T_S$  are valid. Since all leaves are of the form  $D_1, \dots, D_n, \Pi \rightarrow \Sigma$  where  $\Pi$  and  $\Sigma$  are sets of atoms and  $D_1, \dots, D_n$  are definitions, it follows from Theorem 3 that they are provable in **LPC(ID)**. Hence, by (b) of Proposition 5,  $\Gamma \rightarrow \Delta$  is provable in **LPC(ID)**.

## 5 Complexity results

In this section, we provide some complexity results for **PC(ID)**, which may give some helpful insight into the reasoning problems in **PC(ID)**.

**Proposition 6.** *Satisfiability problem in **PC(ID)** is NP-complete.*

*Proof.* (Membership) Propositional well-founded models can be computed in polynomial time, e.g. using the algorithm of Van Gelder in [20]. It is easy to define an algorithm that uses this well-founded semantics algorithm and finds models that satisfy **PC(ID)** theories in polynomial time on a non-deterministic turing machine.

(Hardness) Any satisfiability problem for propositional logic is trivially also a satisfiability problem for **PC(ID)**.

Recall Definition 3 of totality of a definition  $D$  with respect to a theory  $T$ : for each  $I \models \bigwedge T$ , the well-founded model of  $D$  extending  $I|_{\tau_D^o}$  must be two-valued. Deciding totality is an interesting problem, not least because we cannot even formulate an inference rule to prove totality of a propositional inductive definition in the context of a certain set of **PC(ID)**-formulas.

**Proposition 7.** *Deciding whether a given propositional inductive definition is total with respect to a given propositional theory is co-NP-complete problem.*

*Proof.* (Membership) Let  $D$  be a propositional inductive definition,  $T$  a propositional theory. Any interpretation  $I$  such that  $I \models \bigwedge T$  and the well-founded model of  $D$  extending  $I|_{\tau_D^o}$  is not two-valued, is a certificate for the non-totally of  $D$  with respect to  $T$ . Both checking whether  $I \models \bigwedge T$  and whether the well-founded model of  $D$  extending  $I|_{\tau_D^o}$  is two-valued can be done in polynomial time.

(Hardness) Consider the definition  $D = \{ P \leftarrow \neg P \wedge T \}$ .  $D$  is total with respect to the empty theory if and only if  $T$  is unsatisfiable. Thus we have found an instance of our decision problem that is equivalent to a co-NP-hard decision problem, namely unsatisfiability problem for propositional logic.

## 6 Conclusions, related and further work

We presented a deductive system for the propositional fragment of **FO(ID)** which extends the sequent calculus for propositional logic. The main technical results

are the soundness and completeness theorems of **LPC(ID)**. We also provide some complexity results for PC(ID).

Related work is provided by Hagiya and Sakurai in [22]. They proposed to interpret a (stratified) logic program as iterated inductive definitions of Martin-Löf [30] and developed a proof theory which is sound with respect to the perfect model, and hence, the well-founded semantics of logic programming. A formal proof system based on tableau methods for analyzing computation for Answer Set Programming (ASP) was given as well by Gebser and Schaub [18]. As shown in [26], ASP is closely related to FO(ID). The approach presented in [18] furnishes declarative and fine-grained instruments for characterizing operations as well as strategies of ASP-solvers and provides a uniform proof-theoretic framework for analyzing and comparing different algorithms, which is the first of its kind for ASP.

The first topic for future work, as mentioned in Section 1, is the development and implementation of a proof checker for MINISAT(ID). This would require more study on resolution-based inference rules since MINISAT(ID) is basically an adaption of the DPLL-algorithm for SAT [10,9].

On the theoretical level, we plan to develop proof systems and decidable fragments of FO(ID). As mentioned in Section 1, FO(ID) is not even semi-decidable and thus, a sound and complete proof system for FO(ID) does not exist. Therefore, we hope to build useful proof systems for FO(ID) that can solve a broad class of problems and investigate subclasses of FO(ID) for which they are decidable.

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